ASYMPTOTIC SHAPES FOR ERGODIC FAMILIES OF METRICS ON NILPOTENT GROUPS

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ABSTRACT. Let Γ be a finitely generated virtually nilpotent group. We consider three closely related problems: (i) convergence to a deterministic asymptotic cone for an equivariant ergodic family of inner metrics on Γ, generalizing Pansu's theorem; (ii) the asymptotic shape theorem for First Passage Percolation for general (not necessarily independent) ergodic processes on edges of a Cayley graph of Γ; (iii) the sub-additive ergodic theorem over a general ergodic Γ-action. The limiting objects are given in terms of a Carnot-Carathéodory metric on the graded nilpotent group associated to the Mal'cev completion of Γ.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let Γ be a finitely generated virtually nilpotent group. The topic of this paper may be viewed from three slightly different perspectives:

(i) As a generalization of the result of Pansu [12] showing that the asymptotic cone of an invariant inner metric d on Γ is the Carnot group G_∞ (the graded nilpotent Lie group associated with the Mal'cev completion G of Γ) equipped with a certain Carnot-Carathéodory metric d_∞. Here we show that if one replaces a *single invariant* metric d by an *equivariant ergodic family* $\{d_x \mid x \in X\}$ of inner metrics on Γ, then a.e. (Γ, d_x, e) has the same asymptotic cone which is the Carnot group G_∞ equipped with a fixed Carnot-Carathéodory metric associated to certain averages of the family $\{d_x \mid x \in X\}$.

(ii) As a result about asymptotic shape for First Passage Percolation model over Γ driven by a general ergodic process $\Gamma \curvearrowright (X, m)$. (The case of independent times was recently studied by Benjamini and Tessera [2]).

(iii) As a Subadditive Ergodic Theorem over a general ergodic probability measure preserving (hereafter p.m.p.) action $\Gamma \curvearrowright (X, m)$. Given a measurable function $c : \Gamma \times X \to \mathbb{R}$, satisfying
$$c(\gamma_1 \gamma_2, x) \leq c(\gamma_1, \gamma_2.x) + c(\gamma_2, x) \qquad (\gamma_1, \gamma_2 \in \Gamma),$$
and some additional conditions, we show that for a.e. $x \in X$ there is a unique limit to $c(\gamma, x)$ suitably normalized; the limit is described on the Carnot group G_∞ using a Carnot-Carathéodory construction.

Let us recall some facts about nilpotent groups. Upon passing to a finite index subgroup and dividing by a finite normal subgroup, we assume hereafter that our group Γ is a torsion-free nilpotent group with torsion-free abelianization $\Gamma^{ab} \cong \mathbb{Z}^d$; this adjustment does not change the problem - see §2.4 below. By the classical work of Mal'cev, a finitely generated, torsion-free, nilpotent group Γ can be embedded as a discrete subgroup of a connected, simply connected, nilpotent real Lie group G so that G/Γ is compact. Moreover, such an embedding $\Gamma < G$ is unique up to automorphisms of G. This G is often called the **Mal'cev completion** of Γ. Associated with G one has the

graded nilpotent connected, simply connected, real Lie group G_∞, that is constructed from the quotient spaces $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ of the descending central series $\mathfrak{g} = \mathfrak{g}^1 > \mathfrak{g}^2 > \cdots > \mathfrak{g}^{r+1} = \{0\}$ of the Lie algebra of G (see below). In particular, one can identify the abelianizations $G^{\mathrm{ab}} := G/[G, G]$ and $G^{\mathrm{ab}}_\infty = G_\infty/[G_\infty, G_\infty]$ via $\mathfrak{g}/\mathfrak{g}^2 \cong \mathfrak{g}_\infty/\mathfrak{g}^2_\infty$. The graded Lie group G_∞ admits a one parameter family $\{\delta_t \mid t > 0\}$ of automorphisms that induce the linear homotheties $\times t$ on the real vector space $G^{\mathrm{ab}}_\infty \cong \mathfrak{g}^{\mathrm{ab}} \cong \mathfrak{g}^{\mathrm{ab}}_\infty$. Such a group G_∞ (with the family of homotheties) is sometimes called a **Carnot group**.

Example 1.1. The integral Heisenberg group $\mathbf{H}_\mathbb{Z}$ embeds in the 3-dimensional real Heisenberg group

$$\mathbf{H}_\mathbb{R} = \left\{ M_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

by restricting x, y, z to be integers. In this case $G = \mathbf{H}_\mathbb{R}$ is itself graded: $G = G_\infty$. The abelianization G^{ab} is two dimensional, and $G \to G^{\mathrm{ab}}$ is given by $M_{x,y,z} \mapsto (x, y)$. The similarities are given by

$$\delta_t(M_{x,y,z}) = M_{tx,ty,t^2 z}.$$

Let d be a an inner right-invariant[1] metric d on Γ, e.g. $d(\gamma_1, \gamma_2) = |\gamma_1 \gamma_2^{-1}|_S$, where $|\gamma|_S$ is the length of a shortest word representing γ using elements of a fixed generating set S for Γ. In [12] Pansu proved that associated with such d there is a right-invariant proper metric d_∞ on G_∞, that is homogeneous in the sense that

$$d_\infty(\delta_t(g), \delta_t(g')) = t \cdot d_\infty(g, g') \qquad (g, g' \in G_\infty, \ t > 0)$$

and such that there is Gromov-Hausdorff convergence

$$(1.1) \qquad (\Gamma, \frac{1}{t} \cdot d, e) \ \to \ (G_\infty, d_\infty, e).$$

The metric d_∞ is a result of Carnot-Carathéodry construction applied to a certain norm on $G^{\mathrm{ab}} \cong G^{\mathrm{ab}}_\infty$, associated to d.

To state our results we need to fix some further notations. Let Γ be a finitely generated, torsion-free, nilpotent group, denote by G its Mal'cev completion, and by G_∞ the associated Carnot group with homotheties $\{\delta_t \mid t > 0\}$. Fix a right-invariant inner metric d on Γ, e.g. a word metric as above, and let d_∞ on G_∞ be the associated Carnot-Carathéodory metric as in Pansu's theorem.

Given a function $f : \Gamma \to \mathbb{R}$ one can consider an asymptotic cone of its graph in $\Gamma \times \mathbb{R}$, i.e. possible Gromov-Hausdorff limits of

$$\mathrm{Graph}(f) = \{(\gamma, f(\gamma)) \mid \gamma \in \Gamma\} \subset \Gamma \times \mathbb{R}$$

with $(e, 0)$ being the marked point. The functions f that will appear below, will be special in several ways:

(f1) the rescaled graphs $\mathrm{Graph}(f)$ actually have a unique Gromov-Hausdorff limit,
(f2) this limit is given by a graph $\mathrm{Graph}(\Phi)$ of a function $\Phi : G_\infty \to \mathbb{R}$,
(f3) the function $\Phi : G_\infty \to \mathbb{R}$ appears in a Carnot-Carathéodory construction; in particular, it is homogeneous: $\Phi(\delta_t(g)) = t \cdot \Phi(g)$ for $g \in G_\infty$ and $t > 0$.

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[1] One often considers left-invariant metrics; our choice of right-invariance is dictated by our notation for sub-additive cocycles.

The convergence in (f2) implies that

$$t_i^{-1} \cdot f(\gamma_i) \to \Phi(g) \qquad \text{whenever} \qquad \text{scl}_{t_i}(\gamma_i) \to g \in G_\infty,$$

where the latter relates to the Gromov-Hausdorff limit (1.1) with $t_i \to \infty$. Let us say that two functions $f, f' : \Gamma \to \mathbb{R}$ are **asymptotically equivalent** if

$$f(\gamma) - f'(\gamma) = o(|\gamma|_S).$$

Then f satisfies (f1)-(f3) with Φ iff f' does. One might say that Φ is **the unique homogeneous representative of the asymptotic equivalence class** of f (here the uniqueness statement follows from the fact that different homogeneous functions cannot be asymptotically equivalent).

Theorem A.
Let Γ be a finitely generated virtually nilpotent group, $\Gamma \curvearrowright (X, m)$ an ergodic probability measure-preserving action, and $c : \Gamma \times X \to \mathbb{R}_+$ a measurable subadditive cocycle. Assume that

(i) *For some $0 < k \le K < +\infty$ one has $k \cdot |\gamma|_S \le c(\gamma, x) \le K \cdot |\gamma|_S$ for a.e. $x \in X$.*
(ii) *For a.e. $x \in X$ for every $\epsilon > 0$ there is a finite set $F \subset \Gamma$ so that for every $x' \in \Gamma.x$ any $\gamma \in \Gamma$ can be written as $\gamma = \delta_n \cdots \delta_2 \delta_1$ with $\delta_i \in F$ and*

$$c(\delta_1, x') + c(\delta_2, \delta_1.x') + \cdots + c(\delta_n, \delta_{n-1} \cdots \delta_1.x') \le (1 + \epsilon) \cdot c(\gamma, x').$$

Then for a full measure set of $x \in X$ the functions $c(-, x) : \Gamma \to \mathbb{R}$ are asymptotically equivalent to each other and are represented by a unique homogeneous function $\Phi : G_\infty \to \mathbb{R}$, that is obtained in the following construction.

Construction 1.2. Given a subadditive cocycle $c : \Gamma \times X \to \mathbb{R}_+$ over an ergodic action $\Gamma \curvearrowright (X, m)$ of a finitely generated virtually nilpotent group Γ.

- Up to finite index and finite kernel (once $\Gamma \curvearrowright (X, m)$ and $c : \Gamma \times X \to \mathbb{R}_+$ are adjusted accordingly) we are reduced to the case that Γ is a finitely generated nilpotent group that is torsion free and has torsion-free abelianization Γ^{ab}.
- Define a subadditive function $\overline{c} : \Gamma \to \mathbb{R}_+$ by integration:

$$\overline{c}(\gamma) := \int_X c(\gamma, x) \, dm(x).$$

- Define a subadditive function $f : \Gamma^{\mathrm{ab}} \longrightarrow \mathbb{R}_+$ by minimizing F over fibers:

$$f(\gamma^{\mathrm{ab}}) := \inf \left\{ \overline{c}(\gamma_1) \mid \gamma^{\mathrm{ab}} = \gamma_1^{\mathrm{ab}} \right\}.$$

- Define $\phi : \mathfrak{g}_\infty^{\mathrm{ab}} \to \mathbb{R}_+$ by viewing Γ^{ab} as a lattice in the vector space $\Gamma^{\mathrm{ab}} \otimes \mathbb{R}$ and observing that there is a unique homogeneous subadditive function (a possibly asymmetric norm)

$$\phi : \Gamma^{\mathrm{ab}} \otimes \mathbb{R} \longrightarrow \mathbb{R}_+$$

representing $f : \Gamma^{\mathrm{ab}} \longrightarrow \mathbb{R}_+$.
- Define $\Phi : G_\infty \to \mathbb{R}_+$ to be the homogeneous function associated to ϕ viewed as an asymmetric norm on $\Gamma^{\mathrm{ab}} \otimes \mathbb{R} \cong G^{\mathrm{ab}} \cong \mathfrak{g}^{\mathrm{ab}} \cong \mathfrak{g}_\infty^{\mathrm{ab}}$ and applying the Carnot-Carathéodory construction.

For more details see §§2.2–2.4.

Recall that a metric d on a metric space M is called **inner** if given $\epsilon > 0$ there is $R < \infty$ so that for any $p, q \in M$ one can find $n \in \mathbb{N}$ and $p_0, \ldots, p_n$ so that: $p_0 = p$, $p_n = q$, $d(p_{i-1}, p_i) < R$ for $1 \leq i \leq n$, and

$$d(p_0, p_1) + d(p_1, p_2) + \cdots + d(p_{n-1}, p_n) \leq (1 + \epsilon) \cdot d(p, q).$$

The following result can be viewed as a generalization of Pansu's result on a single right-invariant inner metric on Γ to equivariant ergodic families of inner metrics.

Theorem B.
Let Γ be a finitely generated virtually nilpotent group, $\Gamma \curvearrowright (X, m)$ an ergodic p.m.p. action, and let $\{d_x \mid x \in X\}$ be a measurable family of inner metrics on Γ that is right-equivariant:

$$(1.2) \qquad\qquad d_x(\gamma_1, \gamma_2) = d_{\gamma.x}(\gamma_1 \gamma^{-1}, \gamma_2 \gamma^{-1}) \qquad\qquad (\gamma, \gamma_1, \gamma_2 \in \Gamma),$$

and satisfies a uniform bi-Lipschitz estimate $0 < a \leq d_x/d \leq b < \infty$ where d is some right-invariant word metric on Γ.

Then there exists a right-invariant homogeneous metric d_ϕ on G_∞ so that for a.e. $x \in X$ there is Gromov-Hausdorff convergence

$$(\Gamma, \frac{1}{t} \cdot d_x, e) \xrightarrow{GH} (G_\infty, d_\phi, e).$$

Here $d_\phi(g_1, g_2) = \Phi(g_2 g_1^{-1})$ with Φ from Construction 1.2 corresponding to

$$(1.3) \qquad\qquad\qquad c(\gamma, x) := d_x(e, \gamma).$$

One can also start from a sub-additive cocycle $c : \Gamma \times X \to \mathbb{R}_+$ and define

$$(1.4) \qquad\qquad d_x(\gamma_1, \gamma_2) := c(\gamma_2 \gamma_1^{-1}, \gamma_1.x) \qquad (x \in X, \ \gamma_1, \gamma_2 \in \Gamma).$$

The resulting measurable family of functions is equivariant (as in (1.2)), and each is a (possibly asymmetric) metric on Γ; condition A(ii) on c corresponds to d_x being inner.

A natural example of an equivariant family of metrics as above appears in the following setting, known as **First Passage Percolation** model. Fix a Cayley graph (V, E) for Γ defined by some finite symmetric generating set $S \subset \Gamma$ (so $V = \Gamma$ and $E = \{(\gamma, s\gamma) \mid \gamma \in \Gamma, \ s \in S\}$), and fix a $0 < a < b < \infty$. Define $X := [a, b]^E$ – the space of functions $x : E \to [a, b]$; we think of $x_{(v, v')}$ as the time it takes to cross edge $(v, v') \in E$. Since Γ acts by automorphisms on (V, E), it also acts continuously on the compact metric space X. Let m be some Γ-invariant ergodic Borel probability measure on X, e.g. the Bernoulli measure $m = \mu^E$ where μ is some probability measure on $[a, b]$. Every $x \in X$ defines the time it takes to cross any given edge $e \in E$ and we can define

$$d_x(v, v') = \inf \left\{ \sum_{i=1}^n x_{(v_{i-1}, v_i)} \mid v_0 = v, v_n = v', (v_{i-1}, v_i) \in E \right\}$$

to be the minimal travel time from v to v' in the particular realization $x \in X$ of the configuration of passage times of edges. One is now interested in the **asymptotic shape** as $T \to \infty$ of the set

$$\mathrm{B}_x^\Gamma(T) := \{v \in V \mid d_x(e, v) < T\}$$

of vertices that can be reached from the origin $e \in V$ in time $< T$, for a typical configuration $x \in X$.

Corollary C.
With the notations as above, there exists a homogeneous function $\Phi : G_\infty \to \mathbb{R}_+$, given in Construction 1.2, so that for m-a.e. $x \in X$ the sets $\mathrm{B}_x^\Gamma(T)$ are within $o(T)$-approximation from

$$\{g \in G_\infty \mid \Phi(g) < T\}$$

which is a δ_T image of a fixed set:

$$\mathrm{B}_x^\Gamma(T) \;\sim\; \{g \in G_\infty \mid \Phi(g) < T\} = \delta_T\left(\{g \in G_\infty \mid \Phi(g) < 1\}\right).$$

Thus $\{g \in G_\infty \mid \Phi(g) < 1\}$ gives the asymptotic shape of a.e. $\mathrm{B}_x^\Gamma(T)$ rescaled by T for $T \gg 1$.

It follows from Theorem A that for m-a.e. $x \in X$ for any $\epsilon > 0$ for $T > T(x, \epsilon)$

$$\{g \in G_\infty \mid \Phi(g) < 1 - \epsilon\} \;\subset\; \mathrm{scl}_T\left(\mathrm{B}_x^\Gamma(T)\right) \;\subset\; \{g \in G_\infty \mid \Phi(g) < 1 + \epsilon\}$$

which is equivalent to the statement of the Corollary.

Let us make some remarks about these results.

(1) I. Banjamini and R. Tessera [2] recently established the asymptotic shape theorem for the First Passage Percolation model (Corollary C) for the case of an *independent distribution on edges*, i.e. the measure $m = \mu^E$. In this framework their result is stronger: the assumption is weaker (rather than compact support the distribution μ is assumed to have a finite exponential moment) and there is statement of a speed for the convergence to the asymptotic shape. However, the proofs, being based on probabilistic techniques, do not seem to apply to the general ergodic case as in Corollary C.

(2) The abelian case $\Gamma = \mathbb{Z}^d$ was considered by Boivin [4] in the context of First Passage Percolation as in Corollary C, and then by Björkland [3] in the more general context of sub-additive cocycles as in Theorem A. Both results are proved under weaker integrability condition, namely $c(\gamma, -) \in L^{d,1}(X, m)$ (Lorentz space). This integrability condition is known to be sharp for sub-additive cocycles over general ergodic $\mathbb{Z}^d$-actions [5]. We note that in [3] no *a priori* innerness assumption is imposed, but in retrospect it is satisfied.

(3) Assumption (ii) in Theorem A (and the corresponding assumption of innerness of metrics in Theorem B) is necessary for the limit object Φ (and d_ϕ) to be geodesic. Yet, it will become clear from the proof below that this condition is not needed for the inequality

$$\limsup_{\mathrm{scl}_t(\gamma) \to g} \frac{1}{t} \cdot c(\gamma, x) \leq \Phi(g) \qquad (g \in G_\infty)$$

for m-a.e. $x \in X$. In fact, the proof of this inequality (see §4.1) does not require the lower estimate in Theorem A(i); it only uses the inequality $c(\gamma, x) \leq K \cdot |\gamma|_S$, which is equivalent to $c(\gamma, -) \in L^\infty(X, m)$ for $\gamma \in S$ a generating set for Γ.

(4) It is possible that assumption (i) in Theorem A can be relaxed. Yet, note that already in the Abelian case $\Gamma = \mathbb{Z}^d$ pointwise convergence requires $L^{d,1}(X, m)$-integrability.

(5) Let $\Gamma < G$ and G_∞ be as above. Theorems A and B show that asymptotic shapes are classified by Φ (and d_ϕ) for some unique, possibly asymmetric, norm $\phi : \mathfrak{g}_\infty^{\mathrm{ab}} \to \mathbb{R}_+$. The converse also holds: for every asymmetric norm ϕ the associated Carnot-Carathéodory Φ and d_ϕ arise as an asymptotic shape for some cocycle over Γ, in fact from a subadditive function $F : \Gamma \to \mathbb{R}_+$. However, the question of which asymptotic shapes (equivalently norms) can appear in First Passage Percolation with independent distribution on edges remains widely open.

We would like to emphasize the following remark.

Remark 1.3. An important example of subadditive cocycles over group actions are

$$c(\gamma, x) = \log \|A(\gamma, x)\|$$

where $A : \Gamma \times X \to \mathrm{SL}_d(\mathbb{R})$ is a matrix valued cocycle, i.e. satisfies $A(\gamma_1 \gamma_2, x) = A(\gamma_1, \gamma_2.x)A(\gamma_2, x)$. If Γ is not Abelian, then the results of this paper do not apply to such cocycles – they systematically fail the innerness assumption. Yet, for any amenable group Γ (in particular, nilpotent) one can describe the asymptotic behavior of such cocycles: they are asymptotically equivalent to a homogeneous subadditive function, namely the pull-back of a norm ϕ on the abelianization $\Gamma_1^{\mathrm{ab}} \otimes \mathbb{R}$ for some finite index subgroup $\Gamma_1 < \Gamma$. More precisely, the norm has the form

$$\phi(\gamma) = \max_{1 \le j \le d} |\chi_j(\gamma^{\mathrm{ab}})|$$

for some characters $\chi_1, \ldots, \chi_d : \Gamma_1^{\mathrm{ab}} \otimes \mathbb{R} \to \mathbb{R}$. In particular, if Γ is a non-abelian nilpotent group, such homogeneous functions do not grow along the commutator subgroup unlike Carnot-Carathéodry metrics. This can be shown by applying a form of Zimmer's Cocycle Reduction lemma (using the fact that Γ is amenable) that allows one to bring the cocycle to an upper triangular form and read off the growth from the diagonal.

Plan of the paper.
In Section 2 we recall some background on graded nilpotent Lie groups, the Carnot-Carathéodory construction, Pansu's fundamental result on the asymptotic cone of nilpotent groups, and the construction of ϕ, Φ and d_ϕ associated with the sub-additive cocycle $c : \Gamma \times X \to \mathbb{R}_+$. Section 3 contains two basic preliminary results needed for the proofs of the main theorems. One results concerns approximation of admissible curves in the asymptotic cone G_∞ by expressions of the form $T_k^n \cdots T_2^n T_1^n$ that we call *polygonal paths* in Γ. The second result (Theorem 3.3) is of independent interest; it is an ergodic theorem for sub-additive cocycles along above mentioned polygonal paths. With these preparations at hand we prove Theorems A and B in Section 4.

2. The Carnot group as the asymptotic cone

In this section we recall Pansu's construction of the asymptotic cone (G_∞, d_∞) of a finitely generated nilpotent group and give our construction of (G_∞, d_ϕ), the almost sure asymptotic cone of the random (pseudo) metric space (Γ, d_x).

2.1. The graded Lie algebra/group.
Let Γ be a finitely generated, torsion-free, nilpotent group and G be its Mal'cev completion. In this subsection we construct the associated Carnot group. Since the Lie groups here are connected and simply connected, one can work with the Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G, and set

$$\mathfrak{g}^1 := \mathfrak{g}, \qquad \mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i].$$

Being nilpotent, G satisfies $\mathfrak{g}^{r+1} = \{0\}$ for some $r \in \mathbb{N}$. Since $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ (and in particular $[\mathfrak{g}^{i+1}, \mathfrak{g}^j], [\mathfrak{g}^i, \mathfrak{g}^{j+1}] \subset \mathfrak{g}^{i+j+1}$) the Lie bracket on $\mathfrak{g}$ defines a bilinear map

$$\left(\mathfrak{g}^i/\mathfrak{g}^{i+1}\right) \otimes \left(\mathfrak{g}^j/\mathfrak{g}^{j+1}\right) \longrightarrow (\mathfrak{g}^{i+j}/\mathfrak{g}^{i+j+1}),$$

which can then be used to define the Lie bracket $[-, -]_\infty$ on

$$(2.1) \qquad\qquad \mathfrak{g}_\infty := \bigoplus_{i=1}^r \mathfrak{v}_i, \qquad \text{where} \qquad \mathfrak{v}_i := \mathfrak{g}^i/\mathfrak{g}^{i+1}$$

by extending the above maps linearly. The resulting pair $(\mathfrak{g}_\infty, [-,-]_\infty)$ is called the **graded Lie algebra** associated to $\mathfrak{g}$. Note that the linear maps

$$\delta_t : \mathfrak{g}_\infty \to \mathfrak{g}_\infty, \qquad \delta_t(v_1, \ldots, v_r) = (t \cdot v_1, t^2 \cdot v_2, \ldots, t^r \cdot v_r),$$

satisfy $\delta_t([v,w]_\infty) = [\delta_t(v), \delta_t(w)]_\infty$ and $\delta_{ts} = \delta_t \circ \delta_s$ for $v, w \in \mathfrak{g}_\infty$, $t, s > 0$. Hence $\{\delta_t \mid t > 0\}$ is a one-parameter family of automorphisms of the Lie algebra $\mathfrak{g}_\infty$, and therefore define a one-parameter family of automorphisms of the Lie group $G_\infty := \exp_\infty(\mathfrak{g}_\infty)$, that we will still denote by $\{\delta_t \mid t > 0\}$. (Here we denote the exponential map $\mathfrak{g}_\infty \to G_\infty$ by $\exp_\infty$ to distinguish it from $\exp : \mathfrak{g} \to G$).

The graded Lie algebra naturally appears in the following limiting procedure. Choose a splitting of $\mathfrak{g}$ as a direct sum of vector subspaces

$$(2.2) \qquad\qquad \mathfrak{g} = V_1 \oplus \cdots \oplus V_r, \qquad \text{so that} \qquad \mathfrak{g}^i = V_i \oplus \cdots \oplus V_r,$$

and choose a vector space identification $L : \mathfrak{g} \to \mathfrak{g}_\infty$ so that $L(V_i) = \mathfrak{v}_i$ the ith summand of $\mathfrak{g}_\infty$. For $t > 0$ define the vector space automorphism σ_t of $\mathfrak{g}$ by setting $\sigma_t(v) = t^i \cdot v$ for $v \in V_i$ ($i = 1, \ldots, r$). Then the Lie brackets $[-,-]_t$ on $\mathfrak{g}$, given by

$$[v,w]_t := \sigma_{\frac{1}{t}}\left([\sigma_t(v), \sigma_t(w)]\right),$$

defines a Lie algebra structure on $\mathfrak{g}$ that is isomorphic to the original $[-,-] = [-,-]_1$ via σ_t. However, one has

$$[L(v), L(w)]_\infty = \lim_{t \to \infty} [v, w]_t$$

due to the fact that for $v \in V_i$, $w \in V_j$ the "leading term" of $[v,w]$ lies in V_{i+j}, while the higher terms that belong to $V_{i+j+1} \oplus \cdots \oplus V_r$ become insignificant under the rescaling (see [12]).

Using the $\log : G \to \mathfrak{g}$ and $\exp_\infty : \mathfrak{g}_\infty \to G_\infty$ maps we obtain a family of maps

$$(2.3) \qquad\qquad \mathrm{scl}_t(-) : \Gamma \xrightarrow{\leq} G \xrightarrow{\log} \mathfrak{g} \xrightarrow{\sigma_{t^{-1}}} \mathfrak{g} \xrightarrow{L} \mathfrak{g}_\infty \xrightarrow{\exp_\infty} G_\infty \qquad (t > 0)$$

that explains the asymptotic cone description of Pansu [12] as follows. Let d be an inner right-invariant metric d on Γ and

$$(\Gamma, \frac{1}{t} \cdot d, e) \xrightarrow{GH} (G_\infty, d_\infty, e)$$

the Gromov-Hausdorff convergence. Then a sequence $\gamma_i \in \Gamma$, rescaled by t_i^{-1} with $t_i \to \infty$ as $i \to \infty$, converges to $g \in G_\infty$ iff $\mathrm{scl}_{t_i}(\gamma_i) \to g$ in G_∞. We shall often write

$$g = \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i \qquad \text{instead of} \qquad \mathrm{scl}_{t_i}(\gamma_i) \to g.$$

The metric part of the statement shows that for $t_i \to \infty$ and $\gamma_i, \gamma_i' \in \Gamma$

$$(2.4) \qquad g = \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i, \quad g' = \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i' \qquad \Longrightarrow \qquad d_\infty(g, g') = \lim_{i \to \infty} \frac{1}{t_i} \cdot d(\gamma_i, \gamma_i').$$

The limiting distance d_∞ on G_∞ is **homogeneous** in the sense that

$$d_\infty(\delta_s(g), \delta_s(g')) = s \cdot d_\infty(g, g') \qquad (g, g' \in G_\infty, \ s > 0).$$

This distance is right-invariant (this follows from Lemma 2.2). The distance d_∞ appears in the sub-Finsler Carnot-Carathéodory construction discussed below. Meanwhile let us point out two Lemmas.

Lemma 2.1. *For $\gamma \in \Gamma$ one has*

$$\lim_{n \to \infty} \frac{1}{n} \bullet \gamma^n = \exp_\infty(L \circ \pi \circ \log(\gamma)) = \exp_\infty(\pi_\infty \circ L \circ \log(\gamma)),$$

where $\pi : \mathfrak{g} \to V_1$ and $\pi_\infty : \mathfrak{g}_\infty : \mathfrak{g}_\infty \to \mathfrak{v}_1$ are the linear projection corresponding to (2.2), (2.1).

Proof. Denote by $\pi_k : \mathfrak{g} \to V_k$ $(k = 1, \ldots, r)$ the linear projections according to (2.2), so $\pi = \pi_1$. Then

$$\frac{1}{n} \bullet \gamma^n = \exp_\infty \left(\sum_{k=1}^r \frac{1}{n^k} \cdot L \circ \pi_k \circ \log(\gamma^n) \right) = \exp_\infty \left(L \circ \pi_1 \circ \log(\gamma) + \sum_{k=2}^r \frac{1}{n^{k-1}} \cdot L \circ \pi_k \circ \log(\gamma) \right)$$

and, since $n^{-k+1} \cdot L \circ \pi_k \circ \log(\gamma) \to 0$ for $2 \le k \le r$, the statement is clear. $\qquad\square$

Lemma 2.2.
Given sequences $t_i \to \infty$, $\gamma_i, \gamma_i' \in \Gamma$ with $\frac{1}{t_i} \bullet \gamma_i \to g$ and $\frac{1}{t_i} \bullet \gamma_i' \to g'$ then $\frac{1}{t_i} \bullet \gamma_i \gamma_i' \to gg'$.

Proof. This follows from the Baker-Campbell-Hausdorff formula (cf. §3.3 and the proof of Lemma 5.5 in [6]). $\qquad\square$

2.2. Carnot-Carathéodory constructions.

We follow [12, (17)-(20)]. Denote by $\mathfrak{g}_\infty^{\mathrm{ab}}$ the abelianization of the graded Lie algebra $\mathfrak{g}_\infty$. It is isomorphic to the abelianization $\mathfrak{g}^{\mathrm{ab}}$ of $\mathfrak{g}$, and can also be identified with the direct summand $\mathfrak{v}_1$ of $\mathfrak{g}_\infty$:

$$\mathfrak{g}_\infty^{\mathrm{ab}} \cong \mathfrak{g}^{\mathrm{ab}} \cong \mathfrak{v}_1 \; < \; \bigoplus_{i=1}^r \mathfrak{v}_i = \mathfrak{g}_\infty.$$

Vectors in $\mathfrak{v}_1 < \mathfrak{g}_\infty$ are called **horizontal**. A tangent vector $v \in T_g G_\infty$ at $g \in G_\infty$ is horizontal if its right-translate under g^{-1} is in $\mathfrak{v}_1 < \mathfrak{g}_\infty = T_e G_\infty$. Hence the horizontal vectors form a sub-bundle of the tangent bundle $T G_\infty$; this is a totally non-integrable sub-bundle because $\mathfrak{g}_\infty$ is generated as a Lie algebra by $\mathfrak{v}_1$. Let us say that a continuous piecewise smooth curve $\xi : [a, b] \to G_\infty$ whose tangent vectors $\xi'(t)$ are horizontal for Lebesgue a.e. $t \in [a, b]$ are **admissible**. Any two points $g_1, g_2 \in G_\infty$ can be connected by an admissible curve – this follows from total non-integrability of the sub-bundle of horizontal vectors by Chow's theorem.

Let $\phi : \mathfrak{g}_\infty^{\mathrm{ab}} \to \mathbb{R}_+$ be an **asymmetric norm** (or rather a not necessarily symmetric norm), that is assume ϕ satisfies for all $v, w \in \mathfrak{g}_\infty^{\mathrm{ab}}$, $t > 0$, and some $0 < a \le b < \infty$:

(2.5)
$$\phi(v + w) \le \phi(v) + \phi(w),$$
$$\phi(t \cdot v) = t \cdot \phi(v),$$
$$a \cdot \|v\| \le \phi(v) \le b \cdot \|v\|$$

for some reference Euclidean norm $\| - \|$. Such an asymmetric norm ϕ can be used to measure horizontal vectors in $T G_\infty$ by right-translating them back to $\mathfrak{v}_1 < \mathfrak{g}_\infty = T_e G_\infty$. Given a curve $\xi : [\alpha, \beta] \to G_\infty$ as above its ϕ-**length** is defined to be

(2.6)
$$\mathrm{length}_\phi(\xi) := \int_\alpha^\beta \phi(\xi'(t)\xi(t)^{-1}) \, dt.$$

We define the ϕ-**distance** by

$$d_\phi(g_1, g_2) := \inf \left\{ \mathrm{length}_\phi(\xi) \,|\, \xi \text{ is an admissible curve from } g_1 \text{ to } g_2 \right\}.$$

Starting from a fixed Euclidean norm $\| - \|$ on $\mathfrak{g}_\infty^{\mathrm{ab}}$, one obtains the *sub-Riemannian* metric $d_{\|-\|}$ on G_∞, also known as a *Carnot-Carathéodory* metric; it is right-invariant, homogeneous with respect to the homotheties $\{\delta_t \mid t > 0\}$, and defines the usual topology on G_∞.

For a general asymmetric norm $\phi : \mathfrak{g}_\infty^{\mathrm{ab}} \to \mathbb{R}_+$ as in (2.5) we obtain

$$d_\phi : G_\infty \times G_\infty \longrightarrow \mathbb{R}_+,$$

that is a right-invariant, homogeneous, asymmetric metric, bi-Lipschitz to a Carnot-Carathéodory metric:

(2.7)
$$
\begin{aligned}
& d_\phi(g_1 g, g_2 g) = d_\phi(g_1, g_2), \\
& d_\phi(\delta_t(g_1), \delta_t(g_2)) = t \cdot d_\phi(g_1, g_2), \\
& d_\phi(g_1, g_2) \leq d_\phi(g_1, h) + d_\phi(h, g_2), \\
& a \cdot d_{\|-\|}(g_1, g_2) \leq d_\phi(g_1, g_2) \leq b \cdot d_{\|-\|}(g_1, g_2).
\end{aligned}
$$

Being right-invariant d_ϕ is completely determined by the function

$$\Phi : G_\infty \to \mathbb{R}, \qquad \Phi(g) := d_\phi(e, g), \qquad d_\phi(g_1, g_2) = \Phi(g_2 g_1^{-1}).$$

This function Φ is sub-additive, homogeneous, and bi-Lipschitz to a Carnot-Carathéodory norm

(2.8)
$$
\begin{aligned}
& \Phi(\delta_t(g)) = t \cdot \Phi(g), \\
& \Phi(g_1 g_2) \leq \Phi(g_1) + \Phi(g_2), \\
& a \cdot d_{\|-\|}(e, g) \leq \Phi(g) \leq b \cdot d_{\|-\|}(e, g).
\end{aligned}
$$

If ϕ is actually a norm, i.e. $\phi(-v) = v$, then Φ and d_ϕ are also symmetric: $\Phi(g^{-1}) = \Phi(g)$ and $d_\phi(g_1, g_2) = d_\phi(g_2, g_1)$. In this case d_ϕ is a sub-Finsler Carnot-Carathéodory metric on G_∞ defined by the norm ϕ. Hereafter we shall use the term **Carnot-Carathéodory metric** (or just a **CC-metric**) when referring to a possibly asymmetric d_ϕ associated to ϕ as in (2.5). Pansu's metric d_∞ on G_∞, referred to in the previous section, is the Carnot-Carathéodory metric associated to a certain norm on $\mathfrak{g}_\infty$, that itself is determined by the given inner right-invariant metric d on Γ [12]. The proof does not really use the symmetry assumption, so it can be applied almost verbatim to asymmetric norms. The infimum in the definition of $d_\phi(g_1, g_2)$ is achieved by a (unique) curve, that will be called a d_ϕ-**geodesic**. But we shall use this fact only in reference to d_∞ (or the classical $d_{\|-\|}$).

The notion of ϕ-length can be extended to curves $\xi : [0, 1] \to G_\infty$ that are d_∞-**rectifiable**, i.e. ones for which

$$\sup \left\{ \sum_{j=1}^n d_\infty \left(\xi(s_{j-1}), \xi(s_j) \right) \mid n \in \mathbb{N}, \ 0 = s_0 < s_1 < \cdots < s_n = 1 \right\} \ < \ +\infty.$$

Pansu shows ([12]) that such a curve is absolutely continuous, a.e. differentiable on $[0, 1]$, and that its derivative is a.e. horizontal, so the integral (2.6) makes sense. The ϕ-length of such curves can also be defined by

$$\mathrm{length}_\phi(\xi) = \sup \left\{ \sum_{j=1}^n d_\phi \left(\xi(s_{j-1}), \xi(s_j) \right) \mid n \in \mathbb{N}, \ 0 = s_0 < s_1 < \cdots < s_n = 1 \right\}.$$

2.3. From a sub-additive function $F : \Gamma \to \mathbb{R}_+$ to a CC-metric on G_∞.

Consider a sub-additive function $F : \Gamma \to \mathbb{R}_+$ that is bi-Lipschitz to a word metric, i.e. satisfies

$$
\begin{aligned}
F(\gamma_1 \gamma_2) &\leq F(\gamma_1) + F(\gamma_2), \\
a \cdot d(e, \gamma) &\leq F(\gamma) \leq b \cdot d(e, \gamma),
\end{aligned}
\tag{2.9}
$$

for some constants $0 < a \leq b < \infty$. Note that the upper linear bound $F(\gamma) \leq bd(e, \gamma)$ follows automatically from subadditivity and the fact that Γ is finitely generated; so the content of the second assumption is the lower linear bound for $F : \Gamma \to \mathbb{R}_+$.

Such a function induces a subadditive function

$$
f : \Gamma^{\mathrm{ab}} \to \mathbb{R}_+
$$

using the following general construction.

Lemma 2.3. *Let $1 \to \Delta \to \Gamma \to \Lambda \to 1$ be a short exact sequence of groups, and $F : \Gamma \to \mathbb{R}_+$ a subadditive function. Then the function*

$$
f : \Lambda \to \mathbb{R}_+ \qquad \text{defined by} \qquad f(\gamma \Delta) := \inf\{F(\gamma \delta) \mid \delta \in \Delta\}
$$

is subadditive.

Proof. Given $\lambda_1, \lambda_2 \in \Lambda$ and $\epsilon > 0$ choose $\gamma_1, \gamma_2 \in \Gamma$ so that $\lambda_i = \gamma_i \Delta$ and $F(\gamma_i) \leq f(\lambda_i) + \epsilon$ for $i = 1, 2$. Then $\lambda_1 \lambda_2 = \gamma_1 \gamma_2 \Delta$, so

$$
f(\lambda_1 \lambda_2) \leq F(\gamma_1 \gamma_2) \leq F(\gamma_1) + F(\gamma_2) \leq f(\lambda_1) + f(\lambda_2) + 2\epsilon.
$$

Since $\epsilon > 0$ is arbitrary we get $f(\lambda_1 \lambda_2) \leq f(\lambda_1) + f(\lambda_2)$. $\qquad\qquad\square$

Now recall that Γ is a uniform lattice in its Mal'cev completion G. In fact, viewing G as the $\mathbb{R}$-points $G = \mathbf{G}_\mathbb{R}$ of a $\mathbb{Q}$-algebraic group $\mathbf{G}$, we may think of Γ as (commensurable to) $\mathbf{G}_\mathbb{Z}$. Taking the abelianization is a $\mathbb{Q}$-algebraic operation, hence Γ^{ab} is (commensurable to) $\mathbf{G}_\mathbb{Z}^{\mathrm{ab}}$, a lattice in $\mathbf{G}_\mathbb{R}^{\mathrm{ab}} = G^{\mathrm{ab}}$. Hence Γ^{ab}, that is abstractly isomorphic to $\mathbb{Z}^d$, is a lattice in G^{ab}, that is continuously isomorphic to $\mathbb{R}^d$ with $d = \dim \mathfrak{v}_1$. One often writes

$$
G^{\mathrm{ab}} = \Gamma^{\mathrm{ab}} \otimes \mathbb{R}
$$

to emphasize that Γ^{ab} is a lattice in the real vector space G^{ab}.

Lemma 2.4. *Let Λ be a lattice in a finite dimensional real vector space V, and $f : \Lambda \to \mathbb{R}_+$ be a subadditive function. Then there exists a unique homogeneous subadditive function $\phi : V \to \mathbb{R}_+$ so that f is asymptotically equivalent to $\phi|_\Lambda$; in particular*

$$
\phi(\lambda) = \lim_{n \to \infty} \frac{1}{n} f(n\lambda) = \inf_{n \geq 1} \frac{1}{n} f(n\lambda).
$$

Moreover, if $c_1 \leq f(\lambda)/\|\lambda\| \leq c_2$ on Λ, then $c_1 \leq \phi(v)/\|v\| \leq c_2$ on $V \setminus \{0\}$.

This is an easy and well known fact; but see Burago's [7] for much finer results in case of a coarsely geodesic metric.

Remark 2.5. It follows that any subadditive function $f : \mathbb{Z}^d \to \mathbb{R}_+$ is automatically **inner** in the following sense: given $\epsilon > 0$ there is $R < \infty$ so that any $\lambda \in \mathbb{Z}^d$ can be written as $\lambda = \lambda_1 + \cdots + \lambda_n$ with

$$
f(\lambda_i) \leq R \quad (i = 1, \ldots, n), \qquad f(\lambda_1) + \cdots + f(\lambda_n) \leq (1 + \epsilon) \cdot f(\lambda).
$$

Indeed, this is clear for the asymmetric norm $\phi : \mathbb{R}^d \to \mathbb{R}_+$ associated with f in Lemma 2.4, and translates to f by the virtue of the approximation.

Lemma 2.6. *Let* $F : \Gamma \to \mathbb{R}_+$ *be a subadditive function,* $f : \Gamma^{\mathrm{ab}} \to \mathbb{R}_+$ *and* $\phi : \Gamma^{\mathrm{ab}} \otimes \mathbb{R} \to \mathbb{R}_+$ *be defined by Lemmas 2.3 and 2.4. Then for any* $\gamma \in \Gamma$ *one has*

$$\lim_{n\to\infty} \frac{1}{n} F(\gamma^n) \;=\; \inf_{n\geq 1} \frac{1}{n} F(\gamma^n) \;=\; \lim_{n\to\infty} \frac{1}{n} f((\gamma^{\mathrm{ab}})^n) \;=\; \inf_{n\geq 1} \frac{1}{n} f((\gamma^{\mathrm{ab}})^n) \;=\; \phi(\gamma^{\mathrm{ab}})$$

for any $\gamma \in \Gamma$ *with* $\gamma^{\mathrm{ab}} = \gamma[\Gamma, \Gamma]$ *denoting the image in* Γ^{ab}.

Proof. The sequence $a_n = F(\gamma^n)$ is sub-additive, i.e. $a_{n+m} \leq a_n + a_m$ for all $n, m \in \mathbb{N}$; hence a_n/n converges to $\inf a_n/n$. Next we note that Lemmas 2.1 and relation (2.4) imply that whenever $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\gamma_1^{\mathrm{ab}} = \gamma_2^{\mathrm{ab}}$ one has

$$\lim_{n\to\infty} \frac{1}{n} \cdot d(\gamma_1^n, \gamma_2^n) = 0.$$

Since any sub-additive function is automatically Lipschitz with respect to the word metric, it follows that $\lim F(\gamma_1^n)/n = \lim F(\gamma_2^n)/n$. Thus this limit of $F(\gamma^n)/n$ depends only on γ^{ab}, and is easily seen to be $\lim f((\gamma^{\mathrm{ab}})^n)/n$, i.e. $\phi(\gamma^{\mathrm{ab}})$. $\qquad\square$

We can now apply the Carnot-Carathéodory construction to define a (possibly asymmetric) metric d_ϕ on G_∞ by

(2.10) $$d_\phi(g, g') := \inf \left\{ \mathrm{length}_\phi(\xi) \,\middle|\, \xi \text{ is an admissible curve from } g \text{ to } g' \right\}.$$

2.4. **From a cocycle** $c : \Gamma \times X \to \mathbb{R}_+$ **to the CC-metric.**

Let Γ be a finitely generated, virtually nilpotent group, $\Gamma \curvearrowright (X, m)$ an ergodic p.m.p. action, and $c : \Gamma \times X \to \mathbb{R}_+$ a subadditive cocycle with $c(\gamma, -) \in L^\infty(X, m)$ for every $\gamma \in \Gamma$. We start with a couple of remarks about passing to finite index subgroups and dividing by finite kernels.

Let $\Gamma' < \Gamma$ be a subgroup of finite index. The action of Γ' on (X, m) has at most $[\Gamma : \Gamma']$-many ergodic components permuted by the Γ-action. Let $c' : \Gamma' \times X' \to \mathbb{R}_+$ be the restriction of c to one of the Γ'-ergodic components $X' \subset X$. If one shows that there is some function $\Phi : G_\infty \to \mathbb{R}_+$ so that for a.e. $x' \in X'$ the function $c'(-, x') : \Gamma' \to \mathbb{R}_+$ is asymptotically equivalent to Φ, then the same would apply to $c(-, x) : \Gamma \to \mathbb{R}_+$ for a.e. $x \in X$. Indeed choosing representatives $\gamma_1, \ldots, \gamma_n$ for $\Gamma' \backslash \Gamma$ for every $\gamma \in \Gamma$ one can write

$$c(\gamma, x) = c(\gamma' \gamma_i, x) \leq c(\gamma', \gamma_i.x) + \|c(\gamma_i, x)\|_\infty$$

for some $\gamma' \in \Gamma'$; similarly

$$c(\gamma', x) = c(\gamma \gamma_i^{-1}, x) \leq c(\gamma, \gamma_i^{-1}.x) + \|c(\gamma_i^{-1}, x)\|_\infty.$$

Hence $c(\gamma, x)$ is at uniformly bounded distance from $c(\gamma', \gamma_i^{\pm 1}.x)$, and therefore has the same asymptotic behavior.

Let N be a finite normal subgroup of Γ. Then $\Gamma_1 := \Gamma/N$ acts ergodically by p.m.p. transformations on $(X_1, m_1) := (X, m)/N$. A subadditive cocycle $c : \Gamma \times X \to \mathbb{R}_+$ defines $c_1 : \Gamma_1 \times X_1 \to \mathbb{R}_+$ by

$$c_1(\gamma_1, x_1) := \max\{c(\gamma, x) \mid \mathrm{pr}(\gamma) = \gamma_1, \ \mathrm{pr}(x) = x_1\}.$$

Then $c_1 : \Gamma_1 \times X_1 \to \mathbb{R}_+$ is a sub-additive cocycle, and it is within bounded distance from $c(\gamma, x)$.

Furthermore we note that conditions (i) and (ii) of Theorem A pass to c' and c_1 as above. (Condition (ii) for c' is an easy exercise using subadditivity and innerness; the others are immediate.) Hence in the context of Theorem A (and Theorem B) we may assume without loss of generality that Γ itself

is finitely-generated, torsion-free, nilpotent group with torsion-free abelianization Γ^{ab}. Hereafter we shall make this assumption.

Let us define the function

$$\overline{c} : \Gamma \to \mathbb{R}_+ \qquad \text{by setting} \qquad \overline{c}(\gamma) := \int_X c(\gamma, x)\, dm(x).$$

Observe that $\overline{c}$ is a sub-additive function, because sub-additivity of c and Γ-invariance of m imply

$$\overline{c}(\gamma_1 \gamma_2) = \int_X c(\gamma_1 \gamma_2, x)\, dm(x) \leq \int_X c(\gamma_1, \gamma_2 x)\, dm(x) + \int_X c(\gamma_2, x)\, dm(x) = \overline{c}(\gamma_1) + \overline{c}(\gamma_2).$$

Moreover one always has an upper linear estimate

$$\overline{c}(\gamma) \leq K_1 \cdot |\gamma|_S \qquad \text{with} \qquad K_1 := \max\{\overline{c}(s) \mid s \in S\}.$$

The definition of $\overline{c}$ requires only L^1-integrability of the functions $c(\gamma, x)$. We note the point-wise bi-Lipschitz condition (i) passes to the average, and we have

(2.11)
$$k \cdot |\gamma|_S \leq \overline{c}(\gamma) \leq K_1 \cdot |\gamma|_S$$

with constants $0 < k \leq K_1 < +\infty$ and any $\gamma \in \Gamma$.

Remark 2.7. It does not seem to be obvious why the condition of being *inner* for the sub-additive cocycle $c : \Gamma \times X \to \mathbb{R}_+$ (condition (ii) in Theorem A) should imply innerness for the average sub-additive function $\overline{c} : \Gamma \to \mathbb{R}_+$. It will follow from our results that for an L^∞-cocycle c over an *ergodic* Γ-action the average $\overline{c}$ is indeed inner as it is asymptotically equivalent to a Carnot-Carathéodory function Φ.

We can now summarize the construction

Proposition 2.8. *Let $\Gamma \curvearrowright (X, m)$ and $c : \Gamma \times X \to \mathbb{R}_+$ be a subadditive cocycle satisfying condition (i) in Theorem A. Then:*

- *The average function*

$$\overline{c}(\gamma) := \int_X c(\gamma, x)\, dm(x)$$

  is a subadditive function on Γ, satisfying bi-Lipschitz condition (2.11).
- *This subadditive function defines a subadditive, homogeneous $\phi : \Gamma^{\mathrm{ab}} \otimes \mathbb{R} \to \mathbb{R}_+$, such that*

$$\lim_{n \to \infty} \frac{1}{n} \overline{c}(\gamma^n) = \phi(\gamma^{\mathrm{ab}}) \qquad (\gamma \in \Gamma).$$

  Moreover, for some $0 < a \leq b < \infty$ one has $a \cdot \|v\| \leq \phi(v) \leq b \cdot \|v\|$ for all $v \in \Gamma^{\mathrm{ab}} \otimes \mathbb{R} \cong \mathfrak{g}_\infty^{\mathrm{ab}}$.
- *The Carnot-Carathéodory construction defines an asymmetric distance on G_∞*

$$d_\phi : G_\infty \times G_\infty \to \mathbb{R}_+$$

  that is right-invariant, homogeneous, and bi-Lipschitz to d_∞ as in (2.7). We denote

$$\Phi(g) := d_\phi(e, g) \qquad (g \in G_\infty).$$

3. Preparation for the main proofs

In this section we prepare two tools for the proof of the main results. The first tool is a purely geometric fact that allows one to approximate an admissible curve in the asymptotic cone G_∞ of Γ by rescaled sequences of the form $T_k^n \cdots T_2^n T_1^n$ in Γ; we call such sequences *polygonal paths*. The second tool is an ergodic theorem for a sub-additive cocycle *along polygonal paths* over a general ergodic, p.m.p. action of a nilpotent group.

3.1. Approximating curves in G_∞ by polygonal paths in Γ.

This subsection concerns purely geometric aspects of the convergence of Γ to its asymptotic cone G_∞ (and is unrelated to the action $\Gamma \curvearrowright (X, m)$ and the cocycle $c : \Gamma \times X \to \mathbb{R}_+$).

As before, Γ is a finitely generated, torsion-free, nilpotent group with torsion-free abelianization, d is a right-invariant word metric on Γ, (G_∞, d_∞) is the asymptotic cone, and

$$\mathrm{scl}_t\, (-) : \Gamma \longrightarrow G_\infty \qquad (t > 0)$$

are the maps defined in (2.3) that realize the Gromov-Hausdorff convergence

$$(\Gamma, \frac{1}{t} \cdot d, e_\Gamma) \longrightarrow (G_\infty, d_\infty, e).$$

We also fix a (possibly) asymmetric norm

$$\phi : \mathfrak{g}_\infty^{\mathrm{ab}} \longrightarrow \mathbb{R}_+$$

satisfying (2.5) and use it to associate length $\mathrm{length}_\phi(\xi)$ to admissible curves $\xi : [0, 1] \to G_\infty$. We denote the balls in G_∞ by

$$\mathrm{B}(g, \epsilon) := \{g' \in G_\infty \mid d_\infty(g, g') < \epsilon\}.$$

Proposition 3.1 (Approximation of curves by polygonal paths).
Given a Lipschitz curve $\xi : [0, 1] \to G_\infty$ with $\xi(0) = e$, and $\epsilon > 0$ one can find $k, p, n_0 \in \mathbb{N}$, $T_1, \ldots, T_k \in \Gamma$ so that for $n \geq n_0$ one has:

$$\sum_{j=1}^k d_\infty \left(\frac{1}{np} \bullet T_j^n \cdots T_2^n T_1^n, \xi(\frac{j}{k}) \right) < \epsilon,$$

and

$$\left| \frac{1}{p} \cdot \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right) - \mathrm{length}_\phi(\xi) \right| < \epsilon.$$

We emphasize the order of the main quantifiers: the elements $T_1, \ldots, T_k$ and $p \in \mathbb{N}$ depend only on the required accuracy $\epsilon > 0$ (and of course the curve ξ), and provide ϵ-good approximation at *all sufficiently large scales*.

We shall need this proposition (in combination with Theorem 3.3) in two cases:

- In § 4.1 we choose ξ to be a ϕ-geodesic connecting e to some g. In this case ξ is a smooth admissible curve and we are interested in the inequality

$$\frac{1}{p} \cdot \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right) \leq \mathrm{length}_\phi(\xi) + \epsilon = \Phi(g) + \epsilon$$

  while $d_\infty(\frac{1}{np} \bullet T_k^n \cdots T_2^n T_1^n, g) < \epsilon$.

- In § 4.2 we get a Lipschitz curve ξ connecting e to some g. In this case we are interested in the inequality

$$\frac{1}{p} \cdot \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right) \geq \mathrm{length}_\phi(\xi) - \epsilon \geq \Phi(g) - \epsilon$$

  while requiring

$$\sum_{j=1}^{k} d_\infty \left(\frac{1}{np} \bullet T_k^n \cdots T_2^n T_1^n, \xi(\frac{j}{k}) \right) < \epsilon$$

  which is stronger than just $d_\infty(\frac{1}{np} \bullet T_k^n \cdots T_2^n T_1^n, g) < \epsilon$.

Proof of Proposition 3.1.
First we work in G_∞. Our goal is to find $k \in \mathbb{N}$ and horizontal vectors

$$v_1, \ldots, v_k \in \mathfrak{v}_1 \subset \mathfrak{g}_\infty$$

so that, denoting $h_j := \exp_\infty(v_j)$ one has

(3.1) $$\sum_{j=1}^{k} d_\infty(h_j \cdots h_1, \xi(\frac{j}{k})) < \frac{1}{2}\epsilon, \qquad |\sum_{j=1}^{k} \phi(v_j) - \mathrm{length}_\phi(\xi)| < \frac{1}{2}\epsilon.$$

For a fixed $k \in \mathbb{N}$, that we will take to be sufficiently large, we define $v_1, \ldots, v_k$ inductively as follows: set

$$v_1 := \pi_\infty \circ \log_\infty(\xi(\frac{1}{k})), \qquad h_1 := \exp_\infty(v_1).$$

Assuming $v_1, \ldots, v_{j-1}$ were chosen, set

$$v_j := \pi_\infty \circ \log_\infty(\xi(\frac{1}{k})(h_{j-1} \cdots h_1 \xi(0))^{-1}), \qquad h_j := \exp_\infty(v_j).$$

Here $\pi_\infty : \mathfrak{g}_\infty \to \mathfrak{v}_1$ is the linear projection corresponding to the decomposition $\mathfrak{g}_\infty = \oplus_{i=1}^{r} \mathfrak{v}_i$.

Let us now show that by choosing k large enough we can guarantee (3.1). To this end we need the fact ([12, Lemme (18)]) that in the unit ball in G_∞ the "horizontal component" gives an approximation with at most quadratic error. More precisely, there is a constant C_1 so that for all $g \in \mathrm{B}(e, 1)$:

$$d_\infty(g, \exp_\infty \circ \pi_\infty \circ \log_\infty(g)) \leq C_1 \cdot d_\infty(e, g)^2.$$

Hence for large k one has for $j = 1, \ldots, k$:

$$d_\infty \left(h_j \cdots h_1, \xi(\frac{j}{k}) \right) \leq C_1 \cdot d_\infty \left(h_{j-1} \cdots h_1, \xi(\frac{j}{k}) \right)^2$$

$$\leq C_1 \cdot \left(d_\infty \left(h_{j-1} \cdots h_1, \xi(\frac{j-1}{k}) \right) + d_\infty \left(\xi(\frac{j-1}{k}), \xi(\frac{j}{k}) \right) \right)^2 \leq C_2 \cdot \frac{1}{k^2}$$

for some C_2 depending on C_1 and the Lipschitz constant of ξ. Hence for all k large enough

$$\sum_{j=1}^{k} d_\infty \left(h_j \cdots h_1, \xi(\frac{j}{k}) \right) < C_2 \cdot k \cdot \frac{1}{k^2} = \frac{C_2}{k} < \frac{1}{2}\epsilon.$$

The second fact that we want to use is that a Lipschitz curve $\xi : [0, 1] \to G_\infty$ is rectifiable. Therefore

$$\mathrm{length}_\phi(\xi) = \lim_{k \to \infty} \sum_{j=1}^{k} d_\phi \left(\xi(\frac{j-1}{k}), \xi(\frac{j}{k}) \right).$$

One also has a constant C so that

$$\left|d_\phi(g, g') - \phi \circ \pi_\infty \circ \log_\infty(g' g^{-1})\right| \leq C \cdot d_\infty(g, g')^2$$

whenever $g' \in \mathrm{B}(g, 1)$. Thus for all sufficiently large k and for each $j = 1, \ldots, k$, we have

$$\left| d_\phi\left(\xi(\frac{j-1}{k}), \xi(\frac{j}{k})\right) - \phi(v_j) \right| \leq \left| d_\phi\left(h_{j-1}\cdots h_1, \xi(\frac{j}{k})\right) - \phi(v_j) \right|$$
$$+ d_\phi\left(h_{j-1}\cdots h_1, \xi(\frac{j-1}{k})\right) \leq C_3 \cdot \frac{1}{k^2}$$

and the second inequality in (3.1) follows.

We have now found $k \in \mathbb{N}$ and horizontal vectors $v_1, \ldots, v_k \in \mathfrak{v}_1$ satisfying (3.1), and need to find $T_1, \ldots, T_k \in \Gamma$, $p \in \mathbb{N}$, and $n_0 > 0$ as in the Proposition. We need the following

Lemma 3.2.
Given a horizontal vector $v \in \mathfrak{v}_1 < \mathfrak{g}_\infty$ and $\epsilon' > 0$ there exist $\tau \in \Gamma$, $p \in \mathbb{N}$ and n_0 so that

$$|\frac{1}{p}\phi(\tau^{\mathrm{ab}}) - \phi(v)| < \epsilon', \qquad d_\infty(\frac{1}{np} \bullet \tau^n, \exp_\infty(v)) < \epsilon' \qquad (n > n_0)$$

where $\exp_\infty : \mathfrak{g}_\infty \to G_\infty$ is the exponential map on G_∞.

Proof. Since $\frac{1}{p} \bullet \Gamma$ becomes denser and denser in G_∞ as $p \to \infty$, one can find $p \in \mathbb{N}$ and $\gamma \in \Gamma$ so that $\frac{1}{p} \bullet \gamma$ is close to $\exp_\infty(v)$. Recall that

$$\frac{1}{p} \bullet \gamma = \exp_\infty\left(\frac{1}{p} \cdot L \circ \pi_1 \circ \log(\gamma) + \frac{1}{p^2} \cdot L \circ \pi_2 \circ \log(\gamma) + \cdots + \frac{1}{p^r} \cdot L \circ \pi_r \circ \log(\gamma)\right)$$

where $\pi_j : \mathfrak{g} \to V_j = L^{-1}(\mathfrak{v}_j)$ are the linear projections. Since $v \in \mathfrak{v}_1 = L(\pi_1(\mathfrak{g}))$, it follows that $p^{-1} \cdot L \circ \pi_1 \circ \log(\gamma)$ and v are close. Hence we may choose p and γ (to be called τ) so that

$$d_\infty\left(\exp_\infty\left(\frac{1}{p} \cdot L \circ \pi_1 \circ \log(\gamma)\right), \exp_\infty(v)\right) < \epsilon', \qquad \left|\phi(\frac{1}{p} \cdot L \circ \pi_1 \circ \log(\gamma)) - \phi(v)\right| < \epsilon'.$$

Now considering powers $\frac{1}{n} \bullet \gamma^n$ we are done by applying Lemma 2.1. This proves Lemma 3.2. $\qquad\square$

Choose $\epsilon' \in (0, \epsilon/2k)$ small enough to ensure that whenever $h'_1, \ldots, h'_k \in G_\infty$ are ϵ'-close to $h_1, \ldots, h_k$, respectively, one has

$$\sum_{j=1}^{k} d_\infty\left(h'_j \cdots h'_2 h'_1, h_j \cdots h_2 h_1\right) < \frac{1}{2}\epsilon.$$

Let us now apply Lemma 3.2 with $\epsilon' > 0$ as above to obtain elements $\tau_1, \ldots, \tau_k$ and $p_1, \ldots, p_k \in \mathbb{N}$ so that the pairs (τ_j, p_j) satisfy

$$|\frac{1}{p_j}\phi(\tau_j^{\mathrm{ab}}) - \phi(v_j)| < \epsilon' < \frac{\epsilon}{2k}, \qquad d_\infty\left(\frac{1}{np_j} \bullet \tau_j^n, h_j\right) < \epsilon' \qquad (j = 1, \ldots, k).$$

Replacing a pair (τ_j, p_j) by $(\tau_j^q, q \cdot p_j)$ with any $q \in \mathbb{N}$, the above inequalities clearly remain valid. So taking $p := p_1 \cdots p_k$ and replacing (τ_j, p_j) by $(T_j := \tau_j^{p/p_j}, p)$ we get elements $T_1, \ldots, T_k \in \Gamma$ so that for $n \gg 1$

$$d_\infty\left(\frac{1}{np} \bullet T_j^n, h_j\right) < \epsilon' \qquad \text{and} \qquad |\frac{1}{p}\phi(T_j^{\mathrm{ab}}) - \phi(v_j)| < \epsilon' < \frac{\epsilon}{2k} \qquad (j = 1, \ldots, k).$$

In view of Lemma 2.2 we know that for every $j = 1, \ldots, k$

$$d_\infty \left(\frac{1}{np} \bullet (T_j^n \cdots T_1^n), (\frac{1}{np} \bullet T_j^n) \cdots (\frac{1}{np} \bullet T_1^n) \right) \to 0.$$

Thus for all n large enough, we have

$$\sum_{j=1}^{k} d_\infty \left(\frac{1}{np} \bullet T_j^n \cdots T_1^n, h_j \cdots h_1 \right) < \frac{1}{2}\epsilon,$$

while

$$\left| \frac{1}{p} \cdot \sum_{j=1}^{k} \phi(T_j^{\mathrm{ab}}) - \sum_{j=1}^{k} \phi(v_j) \right| < \frac{1}{2}\epsilon.$$

Combined with (3.1) this establishes the required inequalities. This completes the proof of Proposition 3.1.   □

3.2. An Ergodic Theorem along polygonal paths.

The goal of this subsection is to prove the following result that might have an independent interest.

Theorem 3.3 (Ergodic theorem along polygonal paths).
Let Γ be a finitely generated torsion-free nilpotent group with torsion-free abelianization, $\Gamma \curvearrowright (X, m)$ an ergodic p.m.p. action, $c : \Gamma \times X \to \mathbb{R}_+$ a measurable, non-negative, subadditive cocycle with $c(\gamma, -) \in L^\infty(X, m)$ for every $\gamma \in \Gamma$, and $\overline{c}$ and ϕ as above. Then for any $T_1, \ldots, T_k \in \Gamma$ one has m-a.e. and $L^1(X, m)$-convergence

$$\lim_{n\to\infty} \frac{1}{n} \cdot c(T_j^n, T_{j-1}^n \cdots T_1^n x) = \phi(T_j^{\mathrm{ab}})$$

for each $j = 1, \ldots, k$, and consequently

$$\lim_{n\to\infty} \frac{1}{n} \left(c(T_k^n, T_{k-1}^n \cdots T_1^n x) + \cdots + c(T_2^n, T_1^n x) + c(T_1^n, x) \right) = \phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}).$$

The L^1 convergence holds under a weaker assumption: $c(\gamma, -) \in L^1(X, m)$.

The case $k = 1$ was shown by Austin [1] under the weaker assumption that $c(\gamma, -) \in L^1(X, m)$ for every $\gamma \in \Gamma$. For reader's convenience we include a proof.

Theorem 3.4 (Austin [1]).
Let $c : \Gamma \times X \to \mathbb{R}_+$ be a subadditive cocycle with $c(\gamma, -) \in L^1(X, m)$ for every $\gamma \in \Gamma$. Then for any $T \in \Gamma$ one has

$$\lim_{n\to\infty} \frac{1}{n} c(T^n, x) = \phi(T^{\mathrm{ab}})$$

for m-a.e. $x \in X$ and in $L^1(X, m)$.

Proof. Kingman's subadditive ergodic theorem, applied to the sub-additive cocycle $h_n(x) := c(T^n, x)$ over (X, m, T), gives an m-a.e. and L^1 convergence

$$\lim_{n\to\infty} \frac{1}{n} c(T^n, x) = h(x),$$

where $h(x)$ is a measurable T-invariant function, satisfying

$$\int_X h(x)\,dm(x) = \lim_{n\to\infty} \frac{1}{n} \cdot \int_X h_n(x)\,dm(x) = \lim_{n\to\infty} \frac{1}{n}\overline{c}(T^n) = \phi(T^{\mathrm{ab}}).$$

(We used Lemma 2.6 in the last equality). Fix $\gamma \in \Gamma$ and denote $\gamma_n := T^n\gamma T^{-n}$. Since $\gamma_n T^n = T^n\gamma$ we have

(3.2) $\qquad c(T^n, x) - c(\gamma_n^{-1}, \gamma_n T^n.x) \leq c(\gamma_n T^n, x) = c(T^n\gamma, x) \leq c(T^n, \gamma.x) + c(\gamma, x).$

Denote $f_n(x) = n^{-1}(c(\gamma_n, x) + c(\gamma_n^{-1}, \gamma_n T^n.x))$ and observe that since one has $|\gamma_n^{-1}|_S = |\gamma_n|_S = o(n)$ (cf. Breuillard [6, Lemma 5.6])

$$\|f_n\|_1 \leq \frac{K_1}{n} \cdot 2|\gamma_n|_S \to 0, \qquad \text{where} \qquad K_1 = \max_{s\in S} \|c(s, -)\|_1.$$

Thus there is a sequence $n_i \to \infty$ so that $f_{n_i}(x) \to 0$ for m-a.e. $x \in X$. Dividing (3.2) by n, and taking the limit along the subsequence n_i, one obtains

$$h(x) \leq h(\gamma.x).$$

Since this is true a.e. for every $\gamma \in \Gamma$, h is Γ-invariant. By ergodicity it is constant. This constant is $\phi(T^{\mathrm{ab}})$ by integration. $\qquad\square$

In the general case of $k \geq 2$ the term $n^{-1} \cdot c(T_1^n, x)$ converges to $\phi(T_1^{\mathrm{ab}})$ by the above, but dealing with the next terms, such as $n^{-1} \cdot c(T_2^n, T_1^n.x)$, one faces a "moving target" problem. We shall overcome this difficulty by finding regions $Z_2, \ldots, Z_k \subset X$ where $n^{-1}c(T_\ell^n, z) = \phi(T_\ell^{\mathrm{ab}}) + o(n)$ for $z \in Z_\ell$, and perturbing the polygonal path $T_k^n \cdots T_2^n T_1^n$ slightly to make sure to land in the appropriate regions at appropriate times. We need several lemmas.

Lemma 3.5 (Parallelogram inequality)**.**
Given $\alpha, \beta, \tau, \tau' \in \Gamma$ one has

$$\big| c(\tau,\, \alpha.x) - c(\tau',\, \beta.x) \big| \leq K \cdot \big(d(\alpha, \beta) + d(\tau\alpha, \tau'\beta) \big)$$

for $K = \max_{s\in S} \|c(s, -)\|_\infty$.

Proof. Let us write $\beta = \delta\alpha$ and $\tau'\beta = \omega\tau\alpha$, so

$$|\delta| = |\delta^{-1}| = d(\alpha, \beta), \qquad |\omega| = |\omega^{-1}| = d(\tau\alpha, \tau'\beta).$$

Since $\tau' = \omega\tau\delta^{-1}$ we have

$$c(\tau', \beta.x) = c(\omega\tau\delta^{-1}, \beta.x) \leq c(\omega, \tau\alpha.x) + c(\tau, \alpha.x) + c(\delta^{-1}, \beta.x)$$
$$\leq c(\tau, \alpha.x) + K \cdot (|\omega| + |\delta^{-1}|).$$

Conversely

$$c(\tau, \alpha.x) = c(\omega^{-1}\tau'\delta, \alpha.x) \leq c(\omega^{-1}, \tau'\beta.x) + c(\tau', \beta.x) + c(\delta, \alpha.x)$$
$$\leq c(\tau', \beta.x) + K \cdot (|\omega^{-1}| + |\delta|).$$

$\qquad\square$

Lemma 3.6. *Let $T \in \Gamma$, $\delta > 0$, and a measurable subset $E \subset X$ be given. Then the set*

$$E^* := \left\{ x \in X \mid \liminf_{n \to \infty} \frac{\#\{n' < \delta \cdot n \mid T^{n-n'}.x \in E\}}{\delta \cdot n} > 0 \right\}$$

has $m(E^*) \geq m(E)$. Moreover, given $\epsilon > 0$ there is N so that the set

$$E_N^* := \left\{ x \in X \mid \forall n \geq N, \ \frac{\#\{n' < \delta \cdot n \mid T^{n-n'}.x \in E\}}{\delta \cdot n} > 0 \right\}$$

has $m(E_N^*) > m(E) - \epsilon$.

Proof. Given a function $f \in L^1(X, m)$ and integers $1 \leq k < n$ consider the averaged function

$$A_k^n f(x) := \frac{1}{n-k} \cdot \sum_{j=k}^{n-1} f(T^j.x).$$

Birkhoff's pointwise ergodic theorem asserts m-a.e. convergence

$$\lim_{n \to \infty} A_0^n f = \mathbb{E}(f \mid \mathcal{B}^T)$$

to the conditional expectation of f with respect to the the sub-ξ-algebra of T-invariant sets $\mathcal{B}^T$. (The conditional expectation is defined only up to null sets, but so is the above convergence). We observe that since

$$A_0^n f(x) = \frac{k}{n} \cdot A_0^k f(x) + \frac{n-k}{n} \cdot A_k^n f(x),$$

taking $k = \lceil (1-\delta)n \rceil$ with $0 < \delta < 1$ fixed and letting $n \to \infty$, it follows that for m-a.e. $x \in X$

$$\frac{1}{\eta \cdot n} \sum_{\lceil (1-\eta)n \rceil}^{n} f \circ T^j \longrightarrow \mathbb{E}(f \mid \mathcal{B}^T).$$

Applying this to the characteristic function $f = 1_E$ of $E \subset X$, we deduce that for m-a.e. $x \in X$

$$\lim_{n \to \infty} \frac{\#\{\lceil (1-\delta)n \rceil \leq j \leq n \mid T^j.x \in E\}}{\delta \cdot n} = h_E(x),$$

where $h_E := \mathbb{E}(E \mid \mathcal{B}^T)$ is the the conditional expectation of 1_E. Since $0 \leq h_E(x) \leq 1$ a.e. while $\int h_E = m(E)$, it follows that the set $\{x \in X \mid h_E(x) > 0\}$ has measure $\geq m(E)$. Yet the set $\{x \mid h_E(x) > 0\}$ is, up to null sets, precisely E^*. Hence $m(E^*) \geq m(E)$.

For the second statement, note that $\{E_N^*\}$ is an increasing sequence of measurable sets whose union (=limit) is E^*. $\square$

Lemma 3.7 (Small perturbations of polygonal paths)**.**
Given $T_1, \ldots, T_k \in \Gamma$ and $\epsilon > 0$, there is $\delta > 0$ and N so that for all $n \geq N$ we have:

$$\frac{1}{n} \cdot d(T_k^{n-n_k} T_2^{n-n_2} T_1^{n-n_1}, \ T_k^n T_{k-1}^n \cdots T_2^n T_1^n) < \epsilon$$

for any $0 \leq n_1, \ldots, n_k \leq \delta \cdot n$.

This Lemma can also be shown by rescaling and passing to the Gromov-Hausdorff limit in G_∞ and relying on Lemma 2.2. Here we give a more direct argument.

Proof. It suffices to show that for fixed $k \in \mathbb{N}$, $T, T_1, \ldots, T_k \in \Gamma$, $\epsilon' > 0$, there is $\delta > 0$ so that for $n \gg 1$ one has:

$$(3.3) \qquad d(T_k^n \cdots T_2^n T_1^n T^{-m}, T_k^n \cdots T_2^n T_1^n) < \epsilon' \cdot n \qquad (\forall m < \delta \cdot n).$$

Indeed, applying such an argument to $T_{j+1}, \ldots, T_\ell$ and $T = T_j$ with $\epsilon' = \epsilon/k$ we get

$$\frac{1}{n}(T_\ell^n \cdots T_{j+1}^n T_j^{n-n_j} T_{j-1}^{n-n_{j-1}} \cdots T_1^{n-n_1}, T_\ell^n \cdots T_{j+1}^n T_j^n T_{j-1}^{n-n_{j-1}} \cdots T_1^{n-n_1}) < \epsilon',$$

and summing these inequalities over $j = 1, \ldots, \ell - 1$, we get the estimate $\ell \cdot \epsilon' \leq \epsilon$ as required.

To establish (3.3) use the general group-theoretic identity $ba = a[a^{-1}, b]b$ to push terms from the right to the left creating some commutator factors. More precisely

$$T_k^n \cdots T_2^n T_1^n T^{-m} = T_k^n \cdots T_2^n \cdot (T^{-m} \cdot [T^m, T_1^n]) \cdot T_1^n$$
$$= T_k^n \cdots T_3^n (T^{-m} \cdot [T^m, T_2^n] \cdot [T^m, T_1^n] \cdot [[T^m, T_1^n]^{-1}, T_2^n]) T_2^n T_1^n = \ldots$$
$$= (T^{-m}[T^m, T_k^n] \cdots [T^m, T_1^n] \cdots) \cdot T_k^n \cdots T_2^n T_1^n$$

where the expression in the parentheses is a product of $O(k)$ factors each being a higher commutator of the form

$$[\cdots [[T^m, T_{j_1}^n]^{-1}, T_{j_2}^n] \cdots, T_{j_s}^n].$$

We need to show that the word length of the expression in parentheses is $< \epsilon n$, and it suffices to show that each of the $O(k)$-commutator expressions has length $< \epsilon' n$, where ϵ' depends on ϵ and k. Iterated commutators of order s above the nilpotency degree r give identity. For $s \leq r$ one has (cf. [6, Lemma 3.8])

$$(3.4) \qquad [\cdots [T^m, T_{j_1}^n]^{-1}, \cdots, T_{j_s}^n] = [\cdots [T, T_{j_1}], \cdots, T_{j_s}]^{\pm m \cdot n^s}$$

For each one of the finitely many elements $\gamma = [\cdots [T, T_{j_1}], \cdots, T_{j_s}]$ as above, we have

$$|\gamma^p|_S \leq C_\gamma \cdot p^{\frac{1}{s+1}} \qquad (p \geq 1),$$

because such γ lies in the $(s+1)$-term of the lower central series $\Gamma^{s+1} = [\Gamma, \Gamma^s] = [\Gamma, [\Gamma \ldots]]$, and the growth rate on this subgroup is asymptotically scaled by t^{s+1} (recall that in the asymptotic cone G_∞ the homothety δ_t acts by multiplication by t^j on the $\mathfrak{g}^j/\mathfrak{g}^{j+1}$-subspace of $\mathfrak{g}_\infty$). Therefore the length of the elements in (3.4) is bounded by

$$C(m \cdot n^s)^{\frac{1}{s+1}} < C(\delta \cdot n^{s+1})^{\frac{1}{s+1}} = C\delta^{\frac{1}{s+1}} \cdot n$$

which can be made $< 2^{-k}\epsilon$ by choosing $\delta > 0$ small enough. $\qquad \square$

Finally, we are ready for the proof of the Ergodic Theorem along Polygonal Paths.

Proof of Theorem 3.3.
Fix $\epsilon > 0$ and let $\delta > 0$ and N be as in Lemma 3.7.

Choose a small $\eta > 0$ and let $M \in \mathbb{N}$ be large enough so that for each $j = 1, \ldots, k$ the set

$$Y_j := \left\{ y \in X \mid \forall n \geq M : \quad |\frac{1}{n} \cdot c(T_j^n, y) - \phi(T_j^{\mathrm{ab}})| < \epsilon \right\} \qquad \text{has} \qquad m(Y_j) > 1 - \eta.$$

Let $Z_k := Y_k$, and apply Lemma 3.6 with $E = Z_k$ to find $M_k \in \mathbb{N}$ so that the set

$$(Z_k)_{M_k}^* := \left\{ z \in X \mid \forall n > M_k, \quad \frac{\#\{n' < \delta \cdot n \mid T^{n-n'}.z \in Z_k\}}{\delta \cdot n} > 0 \right\}$$

satisfies $m((Z_k)^*_{M_k}) > m(Z_k) - \eta > 1 - 2\eta$. Define $Z_{k-1} := Y_{k-1} \cap (Z_k)^*_{M_k}$, and observe that

$$m(Z_{k-1}) > 1 - 3\eta.$$

One then continues inductively to define $Z_{j-1} := (Z_j)^*_{M_j} \cap Y_{j-1}$ (for $j = k-1, \ldots, 3, 2$), where $M_j \in \mathbb{N}$ is chosen large enough to ensure that the set

$$(Z_j)^*_{M_j} := \left\{ z \in X \mid \forall n > M_j, \; \frac{\#\{n' < \delta \cdot n \mid T^{n-n'}.z \in Z_j\}}{\delta \cdot n} > 0 \right\}$$

has

$$m((Z_j)^*_{M_j}) > m(Z_j) - \eta.$$

The sets $Z_1, Z_2, \ldots, Z_k$ that are defined in this manner satisfy

$$m(Z_1) > m(Z_2) - 2\eta > m(Z_3) - 4\eta > \cdots > m(Z_k) - 2(k-1)\eta > 1 - (2k-1)\eta.$$

Let $N := \max(M, M_1, \ldots, M_k)$. Then for every $n > N$ and every $z \in Z_j$, there is $n_j < \delta \cdot n$ so that $T^{n-n_j}.z \in Z_{j+1}$ and

$$\left| \frac{1}{n} \cdot c(T^n_j, z) - \phi(T^{\mathrm{ab}}_j) \right| < \epsilon.$$

Thus for z in a set Z_1 of size $> 1 - (2k-1)\eta$ and every $n \geq N$, there exist $n_1, \ldots, n_k$ all bounded by $\delta \cdot n$, so that

$$\left| \frac{1}{n} \cdot c(T^n_j, T^{n-n_{j-1}}_{j-1} \cdots T^{n-n_2}_2 T^{n-n_1}_1.z) - \phi(T^{\mathrm{ab}}_j) \right| < \epsilon.$$

Applying Lemma 3.7 we have for each $j = 1, \ldots, k$:

$$d(T^n_j T^{n-n_{j-1}}_{j-1} \cdots T^{n-n_2}_2 T^{n-n_1}_1, T^n_j T^n_{j-1} \cdots T^n_2 T^n_1) < n\epsilon,$$

$$d(T^{n-n_{j-1}}_{j-1} \cdots T^{n-n_2}_2 T^{n-n_1}_1, T^n_{j-1} \cdots T^n_2 T^n_1) < n\epsilon.$$

So by Lemma 3.5 and the Lipschitz property we have

$$(3.5) \qquad \left| c(T^n_j, T^{n-n_{j-1}}_{j-1} \cdots T^{n-n_2}_2 T^{n-n_1}_1.z) - c(T^n_j, T^n_{j-1} \cdots T^n_2 T^n_1.z) \right| < 2Kn\epsilon.$$

Therefore for every $x \in Z_1$ and $n > N$ one has:

$$(3.6) \qquad \left| \frac{1}{n} \cdot c(T^n_j, T^n_{j-1} \cdots T^n_2 T^n_1.x) - \phi(T^{\mathrm{ab}}_j) \right| < (2K+1)\epsilon \qquad (j = 1, \ldots, k).$$

Applying this argument with a sequence of $\eta \to 0$, m-a.e. $x \in X$ would belong to at least one of the sets Z_1, and therefore would satisfy (3.6) for all $n > N(x, \epsilon)$. As $\epsilon > 0$ was arbitrary, this proves that for m-a.e. $x \in X$

$$\lim_{n \to \infty} \frac{1}{n} \cdot c(T^n_j, T^n_{j-1} \cdots T^n_2 T^n_1.x) = \phi(T^{\mathrm{ab}}_j) \qquad (j = 1, \ldots, k)$$

which in turn gives the convergence of the sum over $j = 1, \ldots, k$ to $\phi(T^{\mathrm{ab}}_1) + \cdots + \phi(T^{\mathrm{ab}}_k)$. The L^1-convergence here follows by Lebesgue's Dominated convergence, because under the L^∞-assumption the terms are uniformly bounded.

However, the latter conclusion of L^1-convergence does not require the assumption $c(\gamma, -) \in L^\infty(X, m)$, and holds under the weaker assumption $c(\gamma, -) \in L^1(X, m)$ for $\gamma \in \Gamma$. In the pointwise convergence argument, for every x from a set Z_1 of large measure, for all n large enough we compared the values of the cocycle along a polygonal path with that for a perturbed path (3.5) and used Lemma 3.5 to

show that the values are close. In the L^1-context it is more natural to compare a polygonal path with the average of all perturbations:

$$c(T_j^n, T_{j-1}^n \cdots T_1^n.x) - \frac{1}{(\delta n)^{j-1}} \cdot \sum_{n_{j-1}=0}^{\lfloor \delta n \rfloor} \cdots \sum_{n_1=0}^{\lfloor \delta n \rfloor} c(T_j^n, T_{j-1}^{n-n_{j-1}} \cdots T_1^{n-n_1}.x)$$

and replace Lemma 3.5 by its L^1-version:

$$\int_X |c(\tau, \alpha.x) - c(\tau', \beta.x)| \, dm(x) \leq K_1 \cdot (d(\alpha, \beta) + d(\tau\alpha, \tau'\beta))$$

where $K_1 := \max\{\|c(s, -)\|_1 \mid s \in S\}$. We leave out the rather obvious details for this argument, as it it is not needed here. $\qquad\Box$

4. PROOF OF THEOREMS A, B

Throughout this section Γ, $\Gamma \curvearrowright (X, m)$, and $c : \Gamma \times X \to \mathbb{R}_+$ are as in Theorem A, and

$$\phi : \Gamma^{\mathrm{ab}} \otimes \mathbb{R} \cong \mathfrak{g}_\infty \to \mathbb{R}_+, \qquad \Phi : G_\infty \to \mathbb{R}_+, \qquad d_\phi : G_\infty \times G_\infty \to \mathbb{R}_+$$

are as in Proposition 2.8. We denote by d_∞ the corresponding right-invariant, homogeneous metric on G_∞ that appears in Pansu's Carnot-Carathéodory construction. We denote

$$\mathrm{B}(g, \epsilon) := \{g' \in G_\infty \mid d_\infty(g, g') < \epsilon\}$$

the corresponding balls in G_∞. Consider the functions

$$c^*(g, x) := \lim_{\epsilon \searrow 0} \limsup_{t \to \infty} \sup_{\mathrm{scl}_t(\gamma) \in \mathrm{B}(g, \epsilon)} \frac{1}{t} c(\gamma, x),$$

$$c_*(g, x) := \lim_{\epsilon \searrow 0} \liminf_{t \to \infty} \inf_{\mathrm{scl}_t(\gamma) \in \mathrm{B}(g, \epsilon)} \frac{1}{t} c(\gamma, x).$$

While this is not necessary for our argument, it is impossible to ignore the fact that $c^*(g, -)$ and $c_*(g, -)$ are a.e. constant.

Lemma 4.1. *For each $g \in G_\infty$ the functions $c^*(g, -)$, $c_*(g, -)$ are m-a.e. constants, denoted $c^*(g)$, $c_*(g)$, respectively.*

Proof. For any fixed $g \in G_\infty$ the functions $c_*(g, -), c^*(g, -) : X \to \mathbb{R}_+$ are measurable. Fix $\gamma_0 \in \Gamma$. Then for any $\epsilon > 0$ for all $t > t(g, \gamma_0, \epsilon)$ one has

$$\frac{1}{t} \bullet \gamma \in \mathrm{B}(g, \epsilon) \qquad \implies \qquad \frac{1}{t} \bullet \gamma\gamma_0, \frac{1}{t} \bullet \gamma\gamma_0^{-1} \in \mathrm{B}(g, 2\epsilon).$$

Since for every $x \in X$

$$\frac{1}{t} c(\gamma\gamma_0, x) \leq \frac{1}{t} c(\gamma, \gamma_0.x) + \frac{1}{t} c(\gamma_0, x)$$

it follows that $c^*(g, x) \leq c^*(g, \gamma_0.x)$ and $c_*(g, x) \leq c_*(g, \gamma_0.x)$. Applying the same argument to γ_0^{-1} and $\gamma_0.x$ we observe that $c_*(g, -)$ and $c^*(g, -)$ are measurable Γ-invariant functions. Hence they are a.e. constants, because $\Gamma \curvearrowright (X, m)$ is ergodic. $\qquad\Box$

In the following subsections we shall proceed in the following steps:

(1) Show that $c^*(g) \leq \Phi(g)$ for all $g \in G_\infty$.
(2) Show that $\Phi(g) \leq c_*(g)$ for all $g \in G_\infty$.

(3) The obvious inequality $c_* \leq c^*$ combined with the above implies that $t_i^{-1} \cdot c(\gamma_i, x) \to \Phi(g)$ whenever $\mathrm{scl}_{t_i}(\gamma_i) \to g$ in G_∞. We shall show that for a.e. $x \in X$ the above convergence is uniform over $g \in \mathrm{B}(e, 1)$ and will deduce Theorem A by rescaling.

(4) We will prove Theorem B by combining the ideas of the previous steps.

Let $X_0 \subset X$ be the set of $x \in X$ for which $c^*(g, x) = c^*(g)$, $c_*(g, x) = c_*(g)$, and Theorem 3.3 holds for all $k \in \mathbb{N}$ and every choice of $T_1, \ldots, T_k \in \Gamma$. We imposed countably many condition where each holds m-a.e., therefore $m(X \setminus X_0) = 0$.

4.1. The upper bound: $c^*(g) \leq \Phi(g)$.

Fix $x \in X_0$, and assume, towards contradiction, that there exists $\eta > 0$ and sequences $t_i \to \infty$ and $\gamma_i \in \Gamma$ so that

$$(4.1) \qquad \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i = g, \qquad \text{while} \qquad \frac{1}{t_i} c(\gamma_i, x) > d_\phi(e, g) + \eta.$$

Fix a small $\epsilon > 0$, namely $\epsilon = \eta/(K+3)$, where K is as in Theorem A(i). Choose a ϕ-geodesic $\xi : [0, 1] \to G_\infty$, i.e. a smooth admissible curve such that

$$\xi(0) = e, \qquad \xi(1) = g, \qquad \mathrm{length}_\phi(\xi) = \Phi(g)$$

(we could choose any smooth curve from e to g with $\mathrm{length}_\phi(\xi) < \Phi(g) + \epsilon$ with a sufficiently small $\epsilon > 0$). Applying Proposition 3.1 we find $k \in \mathbb{N}$, elements $T_1, \ldots, T_k \in \Gamma$, and a multiple $p \in \mathbb{N}$ that give ϵ-approximation to the curve ξ. Set

$$n_i := \lfloor \frac{t_i}{p} \rfloor, \qquad S_i := T_k^{n_i} \cdots T_2^{n_i} T_1^{n_i}.$$

Note that

$$\limsup_{i \to \infty} \frac{1}{t_i} \cdot d(S_i, \gamma_i) = \limsup_{i \to \infty} d_\infty(\frac{1}{t_i} \bullet S_i, \frac{1}{t_i} \bullet \gamma_i) = \limsup_{i \to \infty} d_\infty(\frac{1}{t_i} \bullet S_i, g) < \epsilon.$$

Since $c(-, x) : \Gamma \to \mathbb{R}_+$ is K-Lipschitz, we have for all sufficiently large $i \gg 1$.

$$\frac{1}{n_i p} \cdot \sum_{j=1}^k c(T_j^{n_i}, T_{j-1}^{n_i} \cdots T_1^{n_i}.x) \geq \frac{1}{n_i p} \cdot c(S_i, x) > \frac{1}{t_i} \cdot c(S_i, x) - \epsilon$$

$$> \frac{1}{t_i} \cdot c(\gamma_i, x) - K \cdot \epsilon - \epsilon > \Phi(g) + (\eta - (K+1)\epsilon).$$

The above inequalities use sub-additivity, the fact that $n_i p/t_i \to 1$, the Lipschitz property of $c(-, x)$, and the assumption (4.1) that we try to refute. Applying Theorem 3.3 we have

$$\lim_{i \to \infty} \frac{1}{n_i p} \cdot \sum_{j=1}^k c(T_j^{n_i}, T_{j-1}^{n_i} \cdots T_1^{n_i}.x) = \frac{1}{p} \cdot \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right).$$

However, by part (ii) of Proposition 3.1, one also has

$$\frac{1}{p} \cdot \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right) < \mathrm{length}_\phi(\xi) + \epsilon < \Phi(g) + \epsilon.$$

This leads to a contradiction, due to our choice of $\epsilon = \eta/(K+3)$. Thus (4.1) is impossible.

4.2. The lower bound: $c_*(g) \geq \Phi(g)$.

Let us now prove the inequality $c_*(g) \geq \Phi(g)$. Fix $g \in G_\infty$, $x \in X_0$, and assume, towards contradiction, that there exists $\eta > 0$ and sequences $t_i \to \infty$ and $\gamma_i \in \Gamma$ so that

$$(4.2) \qquad \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i = g \qquad \text{while} \qquad \frac{1}{t_i} c(\gamma_i, x) < \Phi(g) - \eta.$$

We take a small $\epsilon > 0$ and an associated finite set $F \subset \Gamma$ as in condition (ii) of Theorem A. Apply the following argument to each γ_i from the sequence satisfying (4.2).

Each γ_i can be written as a product

$$\gamma_i = \delta_{i,s_i} \cdots \delta_{i,2} \delta_{i,1},$$

where $\delta_{i,j} \in F$ for all $1 \leq j \leq s_i$ and

$$\sum_{j=1}^{s_i} c(\delta_{i,j}, \, \delta_{i,j-1} \cdots \delta_{i,1}.x) < (1 + \epsilon) \cdot c(\gamma_i, x).$$

Consider the sequence of points

$$g_{i,j} := \frac{1}{t_i} \bullet \delta_{i,j} \cdots \delta_{i,1} \qquad (j = 1, \ldots, s_i).$$

Define a piecewise d_∞-geodesic curve

$$\xi_i : [0, 1] \longrightarrow G_\infty$$

connecting e to $\frac{1}{t_i} \bullet \gamma_i = g_{i,s_i}$ via the points $g_{i,j}$, which are to be visited at times

$$\xi_i\left(\frac{c_{i,1} + \cdots + c_{i,j}}{c_{i,1} + \cdots + c_{i,s_i}}\right) = g_{i,j}, \qquad \text{where} \qquad c_{i,j} := c(\delta_{i,j}, \, \delta_{i,r-1} \cdots \delta_{i,1}.x).$$

Between these times $\xi_i(-)$ follows an appropriately rescaled g_∞-geodesic. So ξ_i traces in G_∞ the points associated to partial products representing a discrete path from e to γ_i, with time parameter chosen according to the $c_{i,j}$-steps.

The bi-Lipschitz condition for $c(-, x)$ in terms of d (condition (i) in Theorem A), implies that $\xi_i : [0, 1] \to G_\infty$ is a uniformly Lipschitz sequence of maps with $\xi_i(0) = e$. Hence by Arzela-Ascoli, upon passing to a subsequence, we may assume that ξ_i converge (uniformly) to a Lipschitz curve

$$\xi : [0, 1] \longrightarrow G_\infty, \qquad \xi(0) = e, \qquad \xi(1) = g.$$

Since we are working towards a contradiction to (4.2) which holds for sub-sequences, we may assume that $\xi_i \to \xi$ without complicating our notations any further.

With the Lipschitz curve ξ at hand and small $\epsilon > 0$, Proposition 3.1 provides $k \in \mathbb{N}$, $T_1, \ldots, T_k \in \Gamma$ and $p \in \mathbb{N}$ that give ϵ-good approximation for the curve ξ: In particular, for large $i \gg 1$ and $n_i := \lfloor t_i/p \rfloor$ one has

$$\sum_{j=1}^{k} d_\infty\left(\frac{1}{n_i p} \bullet (T_j^{n_i} \cdots T_2^{n_i} T_1^{n_i}), \, \xi\left(\frac{j}{k}\right)\right) < \epsilon.$$

For each $i \in \mathbb{N}$, choose $0 = r_i(0) < r_i(1) < \cdots < r_i(k) = s_i$ so that for $j = 1, \ldots, k$:

$$\frac{c_{i,r_i(j-1)+1} + \cdots + c_{i,r_i(j)}}{c_{i,1} + \cdots + c_{i,s_i}} \longrightarrow \frac{1}{k},$$

and write $\gamma_i = \pi_{i,k} \cdots \pi_{i,2} \pi_{i,1}$ with $\pi_{i,j} := \delta_{i,r_i(j)} \cdots \delta_{i,r_i(j-1)+1}$. Then

$$\pi_{i,j} \cdots \pi_{i,2} \pi_{i,1} = \delta_{i,r_i(j)} \cdots \delta_{i,2} \delta_{i,1} \qquad (j = 1, \ldots, k).$$

We have
$$\xi(\frac{j}{k}) = \lim_{i \to \infty} \xi_i(\frac{j}{k}) = \lim_{i \to \infty} \frac{1}{n_i p} \bullet \pi_{i,j} \cdots \pi_{i,2} \pi_{i,1}.$$
Thus for all large enough i:
$$\sum_{j=1}^{k} d_\infty \left(\frac{1}{n_i p} \bullet T_j^{n_i} \cdots T_2^{n_i} T_1^{n_i}, \frac{1}{n_i p} \bullet \pi_{i,j} \cdots \pi_{i,2} \pi_{i,1} \right) < \epsilon$$
and therefore for all large enough i:
$$\sum_{j=1}^{k} \frac{1}{n_i p} \cdot d \left(T_j^{n_i} \cdots T_2^{n_i} T_1^{n_i}, \pi_{i,j} \cdots \pi_{i,2} \pi_{i,1} \right) < \epsilon.$$
We now apply Lemma 3.5 with
$$\alpha = T_{j-1}^{n_i} \cdots T_1^{n_i}, \qquad \tau = T_j^{n_i}, \qquad \beta = \pi_{i,j-1} \cdots \pi_{i,1}, \qquad \tau' = \pi_{i,j},$$
to deduce that for all large enough i:
$$\sum_{j=1}^{k} \frac{1}{n_i p} \cdot \left| c(T_j^{n_i}, \ T_{j-1}^{n_i} \cdots T_1^{n_i}.x) - c(\pi_{i,j}, \ \pi_{i,j-1} \cdots \pi_{i,1}.x) \right| < 2K\epsilon.$$
For $i \gg 1$ we have
$$\frac{1}{t_i} \cdot c(\gamma_i, x) > \frac{n_i p}{(1+\epsilon) t_i} \cdot \frac{1}{n_i p} \cdot \sum_{r=1}^{s_i} c(\delta_{i,r}, \delta_{i,r-1} \cdots \delta_{i,1}.x)$$
$$\geq (1-\epsilon) \cdot \frac{1}{n_i p} \cdot \sum_{j=1}^{k} \sum_{r=r_i(j-1)+1}^{r_i(j)} c(\delta_{i,r}, \delta_{i,r-1} \cdots \delta_{i,1}.x)$$
$$\geq (1-\epsilon) \cdot \frac{1}{n_i p} \cdot \sum_{j=1}^{k} c(\pi_{i,j}, \ \pi_{i,j-1} \cdots \pi_{i,1}.x)$$
$$> (1-\epsilon) \cdot \left(\frac{1}{n_i p} \cdot \sum_{j=1}^{k} c(T_j^{n_i}, \ T_{j-1}^{n_i} \cdots T_1^{n_i}.x) - 2K\epsilon \right)$$
using sub-additivity of c in the third inequality. Theorem 3.3 gives
$$\lim_{i \to \infty} \frac{1}{n_i p} \cdot \sum_{j=1}^{k} c(T_j^{n_i}, T_{j-1}^{n_i} \cdots T_1^{n_i}.x) = \frac{1}{p} \left(\phi(T_k^{\mathrm{ab}}) + \cdots + \phi(T_1^{\mathrm{ab}}) \right) > \mathrm{length}_\phi(\xi) - \epsilon.$$
Since ξ is only one of many possible admissible curves connecting $\xi(0) = e$ to $\xi(1) = g$ (and most likely is sub-optimal in terms of the ϕ-length), one has
$$\mathrm{length}_\phi(\xi) \geq d_\phi(e, g) = \Phi(g).$$
Therefore we deduce
$$\liminf_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma_i, x) \geq (1-\epsilon) \cdot (\Phi(g) - (2K+1)\epsilon).$$
A choice of small enough $\epsilon > 0$ contradicts (4.2). This proves the claimed inequality
$$\Phi(g) \leq c_*(g).$$

4.3. **Proof of Theorem A.**

The results of the two previous subsections giving $c^*(g) \le \Phi(g) \le c_*(g)$, combined with the trivial inequality $c_*(g) \le c^*(g)$, show

$$c_*(g) = c^*(g) = \Phi(g).$$

Equivalently

(4.3) $$\lim_{\epsilon \searrow 0} \limsup_{t \to \infty} \sup \left\{ |\frac{1}{t} \cdot c(\gamma, x) - \Phi(g)| \ : \ \frac{1}{t} \bullet \gamma \in \mathrm{B}(g, \epsilon) \right\} = 0.$$

We need to prove that for m-a.e. $x \in X$ (or rather every $x \in X_0$) one has

$$\forall \epsilon > 0, \quad \exists R < \infty : \qquad |\gamma|_S \ge R \qquad \Longrightarrow \qquad |c(\gamma, x) - \Phi(\mathrm{scl}_1(\gamma))| < \epsilon \cdot |\gamma|_S.$$

Indeed, if the claim were not true, we could find $\epsilon_0 > 0$ and a sequence $\gamma_n \in \Gamma$ with $|\gamma_n|_S \to \infty$, so that

$$|c(\gamma_n, x) - \Phi(\mathrm{scl}_1(\gamma_n))| \ge \epsilon_0 \cdot |\gamma_n|_S.$$

The sequence

$$g_n := \frac{1}{|\gamma_n|_S} \bullet \gamma_n$$

has

$$\limsup_{n \to \infty} d_\infty(g_n, e) \le 1.$$

Hence $\{g_n \mid n \in \mathbb{N}\}$ is bounded. Since balls in G_∞ are precompact, there is a subsequence γ_{n_i} converging to some $g \in G_\infty$ (in fact $g \in \mathrm{B}(e, 1)$). Denote $t_i := |\gamma_{n_i}|_S$. We note that

$$d_\infty(\frac{1}{t_i} \bullet \gamma_{n_i}, \delta_{\frac{1}{t_i}}(\mathrm{scl}_1(\gamma_{n_i}))) \to 0$$

and therefore

$$\lim_{i \to \infty} \frac{1}{t_i} \cdot \Phi(\mathrm{scl}_1(\gamma_{n_i})) = \lim_{i \to \infty} \Phi(\delta_{\frac{1}{t_i}}(\mathrm{scl}_1(\gamma_{n_i}))) = \Phi(\lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_{n_i}) = \Phi(g).$$

Finally (4.3) implies

$$\frac{1}{t_i}|c(\gamma_{n_i}, x) - t_i \cdot \Phi(g)| \to 0$$

contrary to the assumption. This proves Theorem A.

4.4. **Proof of Theorem B.**

The main claim is that given any $g, g' \in G_\infty$ and sequences $t_i \to \infty$, $\gamma_i, \gamma_i' \in \Gamma$, so that

(4.4) $$\lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i \longrightarrow g, \qquad \lim_{i \to \infty} \frac{1}{t_i} \bullet \gamma_i' = g'$$

one necessarily has for every $x \in X_0$:

$$\lim_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma_i' \gamma_i^{-1}, \gamma_i.x) = d_\phi(g, g').$$

To show this we employ a variant on the upper bound argument §4.1 and on the lower bound argument §4.2. In both of these arguments we use a fixed admissible curve ξ_0 connecting e to g in G_∞, concatenated with an appropriate curve ξ connecting g to g' in G_∞.

Denote by $\xi_1 : [0, 2] \to G_\infty$ the curve that connects e to g' via g:

$$\xi_1(0) = e, \qquad \xi_1(1) = g, \qquad \xi_1(2) = g'; \qquad \xi_1(s+1) = \xi(s).$$

Fix a small $\epsilon > 0$, and apply Proposition 3.1 to ξ_1 to find

$$T_1, \ldots, T_{2k} \in \Gamma, \qquad p \in \mathbb{N},$$

so that

$$(4.5) \qquad \left| \frac{1}{p} \cdot \sum_{j=k+1}^{2k} \phi(T_j^{\mathrm{ab}}) - \mathrm{length}_\phi(\xi) \right| < \epsilon,$$

while for all $n \geq n_0$

$$(4.6) \qquad \sum_{j=1}^{k} d_\infty \left(\frac{1}{np} \bullet T_{k+j}^n \cdots T_2^n T_1^n, \xi(\frac{j}{k}) \right) < \epsilon.$$

Note that the last condition is a trivial consequence of the estimate on

$$\sum_{j=1}^{k} d_\infty \left(\frac{1}{np} \bullet T_j^n \cdots T_2^n T_1^n, \xi_1(\frac{j}{k}) \right) + \sum_{j=1}^{k} d_\infty \left(\frac{1}{np} \bullet T_{k+j}^n \cdots T_2^n T_1^n, \xi_1(\frac{j}{k} + 1) \right),$$

while (4.5) can be obtained from approximating the ϕ-lengths of ξ_1 and ξ_0 by

$$\frac{1}{p} \cdot \sum_{j=1}^{2k} \phi(T_j^{\mathrm{ab}}), \qquad \text{and} \qquad \frac{1}{p} \cdot \sum_{j=1}^{k} \phi(T_j^{\mathrm{ab}})$$

and the obvious relation

$$\mathrm{length}_\phi(\xi) = \mathrm{length}_\phi(\xi_1) - \mathrm{length}_\phi(\xi_0).$$

Next, choosing ξ to be a ϕ-geodesic connecting g to g', and taking $n_i = \lfloor t_i/p \rfloor$, we get

$$\lim_{i \to \infty} d_\infty(\frac{1}{n_i p} \bullet \gamma_i, g) = \lim_{i \to \infty} d_\infty(\frac{1}{n_i p} \bullet \gamma_i', g') = 0$$

and

$$\limsup_{i \to \infty} \frac{1}{n_i p} \cdot d(\gamma_i, T_k^{n_i} \cdots T_1^{n_i}) \leq \epsilon, \qquad \limsup_{i \to \infty} \frac{1}{n_i p} \cdot d(\gamma_i', T_{2k}^{n_i} \cdots T_1^{n_i}) \leq \epsilon.$$

Following the same argument as in §4.1 (using Lemma 3.5 and Theorem 3.3), we have

$$\limsup_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma_i' \gamma_i^{-1}, \gamma_i.x) \leq \limsup_{i \to \infty} \frac{1}{n_i p} \cdot c(T_{2k}^{n_i} \cdots T_{k+1}^{n_i}, T_k^{n_i} \cdots T_1^{n_i}.x) + 2K\epsilon$$

$$\leq \lim_{i \to \infty} \frac{1}{n_i p} \cdot \sum_{j=1}^{k} c(T_{k+j}^{n_i}, T_{k+j-1}^{n_i} \cdots T_1^{n_i}.x) + 2K\epsilon = \frac{1}{p} \cdot \sum_{j=1}^{k} \phi(T_{k+j}^{\mathrm{ab}}) + 2K\epsilon$$

$$< \mathrm{length}_\phi(\xi) + (2K + 1)\epsilon = d_\phi(g, g') + (2K + 1) \cdot \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows the upper bound:

$$\limsup_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma_i' \gamma_i^{-1}, \gamma_i.x) \leq d_\phi(g, g').$$

The lower bound,

$$\liminf_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma_i' \gamma_i^{-1}, \gamma_i.x) \geq d_\phi(g, g')$$

is trivial if $g = g'$. Hence we assume $g \neq g'$ which implies that $|\gamma'_i \gamma_i^{-1}|_S \to \infty$. We now use the innerness assumption (corresponding to condition (ii) in Theorem A). Fix an arbitrary small $\epsilon > 0$ and rewrite $\gamma'_i \gamma_i^{-1}$ as a product of

$$\gamma'_i \gamma_i^{-1} = \delta_{i,s_i} \cdots \delta_{i,1}, \qquad \text{while} \qquad \sum_{r=1}^{s_i} c(\delta_{i,r}, \, \delta_{i,r-1} \cdots \delta_{i,1} \gamma_i.x) < (1 + \epsilon) \cdot c(\gamma'_i \gamma_i^{-1}, \gamma_i.x).$$

where $\delta_{i,j}$ belong to a fixed finite set $F \subset \Gamma$ (depending on ϵ and $x \in X_0$). One then proceeds as in §4.2 to construct a uniformly Lipschitz sequence of piecewise geodesic curves connecting g to $\approx g'$, and to use Arzela-Ascoli to pass to a convergent subsequence that produces a Lipschitz curve

$$\xi : [0,1] \to G_\infty, \qquad \xi(0) = g, \qquad \xi(1) = g'.$$

We are going to concatenate ξ_0 with ξ to get $\xi_1 : [0,2] \to G_\infty$ as before. The long products $\gamma'_i \gamma_i^{-1} = \delta_{i,s_i} \cdots \delta_{i,1}$ can be sub-partitioned so that

$$\gamma'_i \gamma_i^{-1} = \pi_{i,k} \cdots \pi_{i,2} \pi_{i,1}$$

while for $j = 1, \ldots, k$ one has

$$\xi(\frac{j}{k}) = \xi_1(\frac{k+j}{k}) = \lim_{i \to \infty} \frac{1}{t_i} \bullet (\pi_{i,j} \cdots \pi_{i,1} \gamma_i).$$

We now invoke the $T_1, \ldots, T_{2k}$ and $p \in \mathbb{N}$ satisfying (4.5) and (4.6). One has

$$\limsup_{i \to \infty} \sum_{j=1}^{k} \frac{1}{n_i p} \cdot d(T_{k+j}^{n_i} \cdots T_1^{n_i} \gamma_i, \, \pi_{i,j} \cdots \pi_{i,1} \gamma_i) \leq \epsilon$$

The sub-additivity gives

$$\sum_{j=1}^{k} c(\pi_{i,j}, \pi_{i,j-1} \cdots \pi_{i,2} \pi_{i,1} \gamma_i.x) \leq \sum_{r=1}^{s_i} c(\delta_{i,r}, \, \delta_{i,r-1} \cdots \delta_{i,1} \gamma_i.x).$$

Combining these facts, one shows that for a subsequence of the given t_i, γ_i, γ'_i one has:

$$\liminf_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma'_i \gamma_i^{-1}, \, \gamma_i.x) \geq \liminf_{i \to \infty} \frac{1}{1+\epsilon} \cdot \frac{1}{n_i p} \sum_{j=1}^{k} c(\pi_{i,j}, \pi_{i,j-1} \cdots \pi_{i,2} \pi_{i,1}, \, \gamma_i.x)$$

$$\geq (1 - \epsilon) \cdot \left(\liminf_{i \to \infty} \frac{1}{n_i p} \cdot \sum_{j=1}^{k} c(T_{k+j}^{n_i}, \, T_{k+j-1}^{n_i} \cdots T_1^{n_i}.x) - 2K\epsilon \right)$$

$$= (1 - \epsilon) \cdot \left(\frac{1}{p} \cdot \left(\phi(T_{2k}^{\text{ab}}) + \cdots + \phi(T_{k+1}^{\text{ab}}) \right) - 2K\epsilon \right)$$

$$\geq (1 - \epsilon) \cdot (\text{length}_\phi(\xi) - (2K+1)\epsilon) \geq (1 - \epsilon) \cdot (d_\phi(g, g') - (2K+1)\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and any subsequence of t_i, γ_i, γ'_i contains a sub-sub-sequence satisfying the above, it follows

$$\liminf_{i \to \infty} \frac{1}{t_i} \cdot c(\gamma'_i \gamma_i^{-1}, \gamma_i.x) \geq d_\phi(g, g').$$

In view of the $\limsup$ inequality, the lower bound is also proven. As in the proof of Theorem A one can easily deduce that for m-a.e. $x \in X$ for every $\epsilon > 0$ there is $R < \infty$ so that for $|\gamma|_S, |\gamma'|_S > R$ one has

$$|c(\gamma' \gamma^{-1}, \gamma.x) - d_\phi(\text{scl}_1(\gamma), \text{scl}_1(\gamma'))| < \epsilon \cdot \max(|\gamma|_S, |\gamma'|_S).$$

This completes the proof of Theorem B.

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